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Hermitian Gravity and Cosmology

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In an attempt to generalize general relativity, we propose a new Hermitian theory of gravity. Space-time is generalized to space-time-momentum-energy and both the principles of general covariance and equivalence are extended. The theory is endowed with a Hermitian metric on a complex manifold. The Hermitian metric contains, apart from the symmetric metric, an anti-symmetric part, which describes dynamical torsion. The causality structure is changed in a way such that there is a minimal time for events to be in causal contact and a maximal radius for a non-local instantaneous causally related volume. The speed of light can exceed the conventional speed of light in non-inertial frames and accelerations are bounded. We have indications that the theory of Hermitian gravity yields general relativity at large scales and a theory equivalent to general relativity at very small scales, where the momenta and energies are very large. As an example, we study cosmology in Hermitian gravity, where matter is described by two scalar fields. While at late times Hermitian gravity reproduces the standard cosmological FLRW models, at early times it differs significantly: quite generically the Universe of Hermitian cosmology exhibits a bounce where a maximal expansion rate (Ricci curvature) is attained. Moreover, we prove that no cosmological constant is permitted at the classical level within our model of Hermitian cosmology.

I. MOTIVATION

According to Albert Einstein's principle of relativity the laws of physics are independent of system of reference. The principle of equivalence states that an observer cannot tell whether he is accelerating or placed in a gravitational field. If we follow Einstein's principle of relativity closely one could argue that there must be a similar principle of equivalence between rotating observers and observers placed in a torsion field. A torsion field is a gravitational field which causes observers to rotate [1]. Theories of generalized gravity, in which dynamical torsion is present, have been proposed, using the standard principle of covariance. The theory proposed by Moffat [2] has unsatisfactory properties [3][4], which caused many to give up dynamical torsion.

In this article we propose a theory of dynamical torsion, by not only extending the principle of equivalence, but also by extending the principle of covariance. We generalize space-time to space-time-momentum-energy, imposing the reciprocity symmetry as a symmetry between space-time and momentum-energy, as was suggested by Max Born in 1938 [5]. Max Born's original motivation behind the idea that the laws of physics should be invariant under the reciprocity transformation was that position and momentum operators of quantum mechanics obey the reciprocity symmetry transformation. Hence a theory unifying quantum mechanics and gravity should also be invariant under the reciprocity transformation. Our aim is to formulate this new theory, incorporating the reciprocity symmetry and simultaneously satisfying an extended principle of general covariance. We will leave the quantum aspect of the theory for future work.

A. The Reciprocity Principle

According to Max Born the laws of physics are invariant under the reciprocity transformation, which is given by

$$x^\mu \to p^\mu \qquad\qquad p^\mu \to -x^\mu, \tag{1}$$

where x^μ and p^μ are the four vectors $(ct, \vec{x})$ and $(\frac{E}{c}, \vec{p})$, respectively. The components of the angular momentum

$$x_\mu p_\nu - p_\mu x_\nu = M_{\mu\nu}$$

are indeed invariant under the reciprocity transformation. Since torsion couples to angular momentum [1], it seems natural to demand that a new theory describing dynamical torsion should be invariant under the reciprocity transformation.

Note that, when quantizing this new theory, which we leave for future work, the commutation relations from quantum mechanics

$$\hat{x}^\mu \hat{p}_\nu - \hat{p}_\nu \hat{x}^\mu = i\hbar\delta^\mu_\nu \tag{2}$$

are also invariant under the reciprocity relation.

B. General Relativity and the Reciprocity Principle

The theory of general relativity describes our universe at large scales (at the moment we are not considering cosmological issues such as dark matter and dark energy) and it generalizes classical mechanical ideas as orbits,

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instead of wave functions, in order to describe particles. The four dimensional line element

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu \tag{3}$$

is a fundamental notion in the theory of general relativity. It is clear that general relativity and the way distances are determined (3) breaks the reciprocity symmetry (1). Demanding that the theory, unifying quantum mechanics and general relativity, should respect the principle of reciprocity, we can state a four dimensional momentum-energy line element

$$d\sigma^2 = \gamma^{\mu\nu} dp_\mu dp_\nu, \tag{4}$$

which should dominate over the space-time line element (3), whenever the momenta are very large compared to this position length scale.

According to the classical laws the momentum p^μ is given by $m\dot{x}^\mu$, which corresponds to the tangent vector of the path taken. The idea of having a tangent space at each point of the manifold, corresponding to the physical idea of the momentum as tangent vector, is clearly only applicable in the classical realm of physics, when momenta are small compared to distances. For the sake of brevity, Max Born called this scale, at which the theory of general relativity is valid the molar world [5], while he called the "small world", which is described by the momentum energy line element (4) the nuclear world. The world, which lies in between these worlds on the energy-momentum and space-time scales, is familiarly called the quantum world, which is the realm of quantum gravity.

Since general relativity is governed by Einstein's equations

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R - \Lambda g_{\mu\nu} = \kappa T_{\mu\nu},$$

we can state via the principle of reciprocity the reciprocal Einstein equations

$$P^{\mu\nu} - \frac{1}{2} \gamma^{\mu\nu} P - \Lambda' \gamma^{\mu\nu} = \kappa' T'^{\mu\nu},$$

which are supposed to govern the momentum-energy curvature of the nuclear world.

We now have a vague idea of how the theory should behave in certain limits and the principles it should obey, namely the principle of reciprocity and the generalized principle of general covariance. Our goal is now to construct a theory which obeys all these limits and principles. The principle of general covariance suggests that there should exist a space-time-momentum-energy line element that specifies a corresponding space-time-momentum-energy interval, which is absolute in the sense that all observers would agree on it; interpreting the momentum-energy coordinates as coordinates, specifying non inertial frames, all relatively non inertial moving observers should agree on the measured space-time-momentum-energy interval. While our proposal extends the standard covariance principle, proposed by Einstein in 1905 [6], it necessarily breaks this principle of covariance by introducing energy-momentum into the distance measurements. Space-time then becomes a relative space with respect to observers moving non inertially with respect to each other and becomes absolute only in the limit of relatively inertial moving observers.

II. ALMOST COMPLEX STRUCTURE

Clearly we need a metric on a manifold to mathematically describe space-time-momentum-energy curvature. In order to build a theory which is reciprocal in momentum and space and at the same time reduces to the theory of general relativity in the molar limit, we need complex manifolds with a Hermitian metric. This is the case because the Hermitian metric is defined such that it is invariant under the reciprocity transformation (1). Any $2d$ dimensional manifold, with a d x and d y coordinates, locally admits a tensor field J [7], which maps the tangent space of the manifold into itself, $J_p : T_pM \to T_pM$, in the following manner

$$J(x^\mu) \to y^\mu \qquad J(y^\mu) \to -x^\mu, \tag{5}$$

where the index μ runs from 0 to $d-1$. It is clear that this map is equivalent to the reciprocity transformation (1), if the y coordinate is interpreted as the energy-momentum coordinate. The map J, also known as the "almost complex structure" operator, may be defined globally on a complex manifold and then it specifies completely the complex structure of the manifold. A metric C, which is invariant under the action of this J operator in the following way

$$C_p(J_pZ, J_pW) = C_p(Z, W) \;,$$

is a Hermitian metric, where $Z, W \in T_pM$ and T_pM is the complexified tangent space [7]. The action of the almost complex structure operator on the basis vectors of the complexified tangent space follows from the definitions of the almost complex structure map and these basis vectors

$$J_p\left(\frac{\partial}{\partial z^\mu}\right) = i\frac{\partial}{\partial z^\mu} \qquad J_p\left(\frac{\partial}{\partial \bar{z}^\mu}\right) = -i\frac{\partial}{\partial \bar{z}^\mu}. \tag{6}$$

Consider a full complex metric

$$\begin{aligned} C = C_{\mu\nu} dz^\mu \otimes dz^\nu \;+\; C_{\mu\bar{\nu}} dz^\mu \otimes dz^{\bar{\nu}} \\ + C_{\bar{\mu}\nu} dz^{\bar{\mu}} \otimes dz^\nu \;+\; C_{\bar{\mu}\bar{\nu}} dz^{\bar{\mu}} \otimes dz^{\bar{\nu}} \,. \end{aligned} \tag{7}$$

A Hermitian metric is a complex metric which has – as a consequence of the reciprocity symmetry – vanishing $C_{\mu\nu}$ and $C_{\bar{\mu}\bar{\nu}}$ components:

$$C = C_{\mu\bar{\nu}} dz^\mu \otimes dz^{\bar{\nu}} + C_{\bar{\mu}\nu} dz^{\bar{\mu}} \otimes dz^\nu,$$

where barred indices $z^{\bar{\mu}} \equiv \bar{z}^\mu$ denote complex conjugation. The basic definitions of complex manifolds do not

differ too much from the usual definitions of a manifold, except for the fact that the complex manifold is locally homeomorphic to the complex space $\mathbb{C}^m$ and the coordinate transformations are holomorphic and hence satisfy the Cauchy-Riemann equations.

III. THE HERMITIAN METRIC

We can write the Hermitian line element in eight dimensional form

$$\begin{aligned} ds^2 &= d\boldsymbol{z}^T \cdot \boldsymbol{C} \cdot d\boldsymbol{z} = d\boldsymbol{z}^m \boldsymbol{C}_{mn} d\boldsymbol{z}^n \\ &= (dz^\mu, dz^{\bar\mu}) \begin{pmatrix} 0 & C_{\mu\bar\nu} \\ C_{\bar\mu\nu} & 0 \end{pmatrix} \begin{pmatrix} dz^\nu \\ dz^{\bar\nu} \end{pmatrix}, \end{aligned} \tag{8}$$

where the Latin indices can take the values $0, 1, ..., d-1, \bar 0, \bar 1, ..., \overline{d-1}$, where the Greek indices can take the values $0, 1, ..., d-1$ and where the number d is the complex dimension of the complex manifold. The entries of the metric C_{mn} are functions of holomorphic and anti antiholomorphic vielbeins defined as follows [1]

$$\begin{aligned} C_{\mu\bar\nu} &= e(z)_\mu{}^a \eta_{ab} e(\bar z)_{\bar\nu}{}^b \\ C_{\bar\mu\nu} &= e(\bar z)_{\bar\mu}{}^a \eta_{ab} e(z)_\nu{}^b, \end{aligned} \tag{9}$$

where $\eta_{ab} \equiv \mathrm{diag}(-1, 1, 1, 1)$. For completeness, we also quote the other two elements of the full complex metric (7),

$$\begin{aligned} C_{\mu\nu} &= e(z)_\mu{}^a \eta_{ab} e(z)_\nu{}^b \\ C_{\bar\mu\bar\nu} &= e(\bar z)_{\bar\mu}{}^a \eta_{ab} e(\bar z)_{\bar\nu}{}^b . \end{aligned} \tag{10}$$

Note that holomorphy of the vielbeins $e^a_\mu = e^a_\mu(z)$ implies the reality condition for the line element,

$$ds^{2\dagger} = ds^2, \tag{11}$$

since it implies the following definition for complex conjugation of the metric,

$$C_{\bar\mu\nu}{}^* \equiv \overline{C_{\bar\mu\nu}} = C_{\mu\bar\nu}.$$

We can also write the Hermitian line element in its familiar four dimensional form

$$ds^2 = 2dz^T \cdot C \cdot d\bar z = 2dz^\mu C_{\mu\bar\nu} dz^{\bar\nu}, \tag{12}$$

which is a logical extension of the complex inner product $\langle w, v\rangle = \bar w_i v_i$. Note that, in this familiar form (12), the Hermitian metric is actually Hermitian in the usual way, $C = C^\dagger$, since the line element is real. We shall see that the eight dimensional notation (8), through which the Hermitian metric takes its symmetric form, $\boldsymbol{C}^T = \boldsymbol{C}$, is very handy for obtaining the equations of motion for this theory. We can define the z^μ and $\bar z^{\bar\mu}$ coordinates in terms of x^μ and $y^{\check\mu}$, as follows[2]

$$z^\mu = \frac{1}{\sqrt 2}(x^\mu + iy^{\check\mu}) \qquad \frac{\partial}{\partial z^\mu} = \frac{1}{\sqrt 2}\Big(\frac{\partial}{\partial x^\mu} - i\frac{\partial}{\partial y^{\check\mu}}\Big)$$

and their complex conjugates. This implies the following decomposition of complex vielbeins in their real, ${e_R}^a_\mu$, and imaginary, ${e_I}^a_{\check\mu}$, parts in the following manner

$$\begin{aligned} e^\mu_a &= {e_R}^\mu_a + i{e_I}^{\check\mu}_a, \qquad \overline{e^\mu_a} = e^{\bar\mu}_a = {e_R}^\mu_a - i{e_I}^{\check\mu}_a \\ e^a_\mu &= {e_R}^a_\mu - i{e_I}^a_{\check\mu}, \qquad \overline{e^a_\mu} = e^a_{\bar\mu} = {e_R}^a_\mu + i{e_I}^a_{\check\mu}. \end{aligned} \tag{13}$$

Vielbeins are holomorphic functions, and thus transform as holomorphic vectors (we consider only the transformation of the Greek indices for this purpose),

$$\begin{aligned} e^a_\mu(z^\nu) &\to \tilde e^a_\mu(w^\nu) = \frac{\partial z^\alpha(w^\nu)}{\partial w^\mu} e^a_\alpha(z^\rho) \\ e^\mu_a(z^\nu) &\to \tilde e^\mu_b(w^\nu) = \frac{\partial w^\mu(z^\nu)}{\partial z^\alpha} e^\alpha_b(z^\rho). \end{aligned} \tag{14}$$

The holomorphy of vielbeins,

$$\partial/\partial z^{\bar\mu} e^a_\nu = 1/\sqrt 2[\partial/\partial x^\mu + i\partial/\partial y^{\check\mu}][{e_R}^a_\nu + i{e_I}^a_\nu] = 0,$$

then implies the Cauchy-Riemann equations,

$$\frac{\partial {e_R}^a_\nu}{\partial x^\mu} = \frac{\partial {e_I}^a_\nu}{\partial y^{\check\mu}}, \qquad \frac{\partial {e_I}^a_\nu}{\partial x^\mu} = -\frac{\partial {e_R}^a_\nu}{\partial y^{\check\mu}}. \tag{15}$$

The Cauchy-Riemann equations (15) then imply that, as a consequence of holomorphy, the tetrads are effectively functions of *four* independent coordinates, even though they are defined on an *eight* dimensional manifold. Thanks to the holomorphy symmetry, the number of physical degrees of freedom of our eight dimensional theory is reduced to that of a four dimensional theory, as required by observations. Conversely, the knowledge of a complex tetrad (both the real and imaginary parts of the tetrad must be known) projected onto the $y^{\check\mu} = 0$ hypersurface $e^a_\mu(x^\nu, 0)$ plus the holomorphy symmetry allows for the unique reconstruction of the full eight dimensional dynamics. This feat is achieved by the simple replacement:

$$e^a_\mu(x^\nu, 0) \to e^a_\mu(\sqrt 2 z^\nu, 0) \equiv e^a_\mu(z^\nu).$$

In this sense our Hermitian gravity is a *holographic* theory. [3] Note that this is true only when tetrads are both complex and holomorphic.

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[1] Here the Latin indices a, b run from $0, 1, ..., d-1$, since they represent local indices, and $\eta = \mathrm{diag}(-1, 1, 1, 1)$. We will the discuss the meaning of these indices later.

[2] Checks (ˇ) are put on indices to denote the imaginary part of a coordinate or on indices of objects, which are projected onto its basis vector.

[3] This is of course quite different from 't Hooft's *holographic* principle [8] for quantum gravity.

Through the inverse relations for the vielbein $e^{\mu}_{a}e^{a}_{\nu} = \delta^{\mu}_{\nu}$ we obtain $C^{\mu\bar{\epsilon}}C_{\bar{\epsilon}\nu} = \delta^{\mu}_{\nu}$, which is in eight dimensional notation equivalent to

$$\boldsymbol{C}^{me}\boldsymbol{C}_{en} = \boldsymbol{\delta}^{m}_{n}. \tag{16}$$

We can now rotate the line element from $z^{\mu}, \bar{z}^{\bar{\mu}}$ space to $x^{\mu}, y^{\check{\mu}}$ space, obtaining

$$\begin{aligned} ds^2 &= d\boldsymbol{x}^T \cdot \boldsymbol{g} \cdot d\boldsymbol{x} = d\boldsymbol{x}^m \boldsymbol{g}_{mn} d\boldsymbol{x}^n \\ &= (dx^{\mu}, dy^{\check{\mu}}) \begin{pmatrix} g_{\mu\nu} & g_{\mu\check{\nu}} \\ g_{\check{\mu}\nu} & g_{\check{\mu}\check{\nu}} \end{pmatrix} \begin{pmatrix} dx^{\nu} \\ dy^{\check{\nu}} \end{pmatrix}, \end{aligned} \tag{17}$$

where the Latin indices take the values $0, 1, ..., d-1, \check{0}, \check{1}, ..., \check{d} - \check{1}$, where the Greek indices take only the values $0, 1, ..., d-1$ and where the number d is again the complex dimension of our manifold. Note that the equation for the eight dimensional inverse (16) holds also for the rotated metric, since its a tensorial equation. We can express the rotated Hermitian metric components, $\boldsymbol{g}_{mn}$, in terms of complex metric components, in terms of real and imaginary parts of the vielbein and in terms of the real symmetric metric $g_{\mu\nu}$ and real anti-symmetric torsion field $B_{\mu\nu}$, in the following manner

$$\begin{aligned} &\frac{1}{2}\begin{pmatrix} (C_{\bar{\mu}\nu} + C_{\mu\bar{\nu}}) & i(C_{\bar{\mu}\nu} - C_{\mu\bar{\nu}}) \\ i(-C_{\bar{\mu}\nu} + C_{\mu\bar{\nu}}) & (C_{\bar{\mu}\nu} + C_{\mu\bar{\nu}}) \end{pmatrix} = \begin{pmatrix} g_{\mu\nu} & B_{\mu\nu} \\ -B_{\mu\nu} & g_{\mu\nu} \end{pmatrix} \\ &= \begin{pmatrix} ({e_R}^{a}_{\mu}{e_R}^{b}_{\nu} + {e_I}^{a}_{\check{\mu}}{e_I}^{b}_{\check{\nu}})\eta_{ab} & (-{e_I}^{a}_{\check{\mu}}{e_R}^{b}_{\nu} + {e_R}^{a}_{\mu}{e_I}^{b}_{\check{\nu}})\eta_{ab} \\ ({e_I}^{a}_{\check{\mu}}{e_R}^{b}_{\nu} - {e_R}^{a}_{\mu}{e_I}^{b}_{\check{\nu}})\eta_{ab} & ({e_R}^{a}_{\mu}{e_R}^{b}_{\nu} + {e_I}^{a}_{\check{\mu}}{e_I}^{b}_{\check{\nu}})\eta_{ab} \end{pmatrix}. \end{aligned}$$

Note that we have defined the eight dimensional rotated Hermitian metric to be symmetric, $\boldsymbol{g} = \boldsymbol{g}^T$. The rotated Hermitian line element is then given by

$$ds^2 = g_{\mu\nu}\left(dx^{\mu}dx^{\nu} + dy^{\mu}dy^{\nu}\right) + 2B_{\mu\nu}dx^{\mu}dy^{\nu}. \tag{18}$$

Clearly this line element is equal to the Hermitian line element in its familiar form (12), when using Einstein's decomposition of the Hermitian line element,

$$C_{\bar{\mu}\nu} = g_{\mu\nu} + iB_{\mu\nu}, \tag{19}$$

which basically rotates the Hermitian line element to $x^{\mu}, y^{\check{\mu}}$ space [9]. Note that this decomposition exhibits Hermiticity explicitly, since g is real and symmetric and B is real and anti-symmetric. The inverse rotated metric can be expressed in terms of inverse vielbeins in the following manner

$$\boldsymbol{g}^{mn} = \begin{pmatrix} {e_R}^{\mu}{e_R}^{\nu} + {e_I}^{\check{\mu}}{e_I}^{\check{\nu}} & -{e_I}^{\check{\mu}}{e_R}^{\nu} + {e_R}^{\mu}{e_I}^{\check{\nu}} \\ {e_I}^{\check{\mu}}{e_R}^{\nu} - {e_R}^{\mu}{e_I}^{\check{\nu}} & {e_R}^{\mu}{e_R}^{\nu} + {e_I}^{\check{\mu}}{e_I}^{\check{\nu}} \end{pmatrix}, \tag{20}$$

where we have suppressed local indices, a and b.

Finally, there is a very handy way of looking at the real and imaginary parts of the vielbein, which allows us to derive the just stated objects and their relations in a very trivial manner. Consider the following holomorphic coordinates $w^{\mu}(z^{\gamma}) = \frac{1}{\sqrt{2}}(u^{\mu} + iv^{\check{\mu}})$ and $z^{\nu}(w^{\delta}) = \frac{1}{\sqrt{2}}(x^{\nu} + iy^{\check{\nu}})$, living in over lapping coordinate patches on a complex manifold. We can perform a coordinate transformation

$$\begin{pmatrix} du^{\mu} \\ dv^{\check{\mu}} \end{pmatrix} = \begin{pmatrix} \frac{\partial u^{\mu}}{\partial x^{\nu}} & \frac{\partial u^{\mu}}{\partial y^{\check{\nu}}} \\ \frac{\partial v^{\check{\mu}}}{\partial x^{\nu}} & \frac{\partial v^{\check{\mu}}}{\partial y^{\check{\nu}}} \end{pmatrix} \begin{pmatrix} dx^{\nu} \\ dy^{\check{\nu}} \end{pmatrix},$$

where there are 32 independent components in the transformation matrix, because of the Cauchy-Riemann equations

$$\frac{du^{\mu}}{dx^{\nu}} = \frac{dv^{\check{\mu}}}{dy^{\check{\nu}}} \qquad \frac{\partial u^{\mu}}{\partial y^{\check{\nu}}} = -\frac{\partial v^{\check{\mu}}}{\partial x^{\nu}}.$$

We can identify the components of the transformation matrix with components of vielbeins, when the set (u, v) forms an orthonormal basis, whenever the (x, y) set forms a coordinate basis. The Cauchy-Riemann equations in terms of vielbeins then become[4]

$${e_R}^{\mu}_{a} \equiv e^{\mu}_{\alpha} = e^{\check{\mu}}_{\check{\alpha}} \qquad {e_I}^{\check{\mu}}_{a} \equiv e^{\check{\mu}}_{\alpha} = -e^{\mu}_{\check{\alpha}}. \tag{21}$$

These definitions of the real and imaginary parts of the vielbeins are consistent with the definition of the decomposition of the complex vielbein in its real and imaginary parts (13). With these definitions the derivation of the rotated metric tensor is immediate. Since in general relativity the metric is defined in terms of vielbeins as follows

$$\boldsymbol{g}^{mn} = \boldsymbol{e}^{m}_{k}\boldsymbol{\eta}^{kl}\boldsymbol{e}^{n}_{l},$$

the $\mu\check{\nu}$ component becomes

$$g^{\check{\mu}\nu} = e^{\check{\mu}}_{k}\boldsymbol{\eta}^{kl}e^{\nu}_{l} = e^{\check{\mu}}_{\kappa}\eta^{\kappa\lambda}e^{\nu}_{\lambda} + e^{\check{\mu}}_{\check{\kappa}}\eta^{\check{\kappa}\check{\lambda}}e^{\nu}_{\check{\lambda}},$$

when defining the rotated flat metric to be $\boldsymbol{\eta} = \mathrm{diag}(-1, 1, 1, 1, -1, 1, 1, 1)$. Using the definitions of the real and imaginary parts of the vielbeins (21), the metric component $g^{\check{\mu}\nu}$ is then given by

$$e^{\check{\mu}}_{\kappa}\eta^{\kappa\lambda}e^{\nu}_{\lambda} - e^{\mu}_{\kappa}\eta^{\kappa\lambda}e^{\check{\nu}}_{\lambda} = ({e_I}^{\check{\mu}}_{a}{e_R}^{\nu}_{b} - {e_R}^{\mu}_{a}{e_I}^{\check{\nu}}_{b})\eta^{ab},$$

which is consistent with the metric components (20) given earlier. We can now state the inverse relations for the real and imaginary components of the vielbeins. In ordinary eight dimensional relativity

$$\boldsymbol{e}^{m}_{k}\boldsymbol{e}^{k}_{l} = \boldsymbol{\delta}^{m}_{l}.$$

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[4] We can prove this as follows. $\langle \hat{w}^{\mu}; \frac{\partial}{\partial \bar{z}^{\beta}} \rangle = \frac{1}{2}\langle \hat{u}^{\mu} + i\hat{v}^{\check{\mu}}; \frac{\partial}{\partial x^{\beta}} + i\frac{\partial}{\partial y^{\beta}} \rangle = \frac{1}{2}\langle e^{\mu}_{\alpha}dx^{\alpha} + e^{\mu}_{\check{\alpha}}dy^{\check{\alpha}} + i(e^{\check{\mu}}_{\alpha}dx^{\alpha} + e^{\check{\mu}}_{\check{\alpha}}dy^{\check{\alpha}}); \frac{\partial}{\partial x^{\beta}} + i\frac{\partial}{\partial y^{\beta}} \rangle = \frac{1}{2}[e^{\mu}_{\beta} + i(e^{\mu}_{\check{\beta}} + e^{\check{\mu}}_{\beta}) - e^{\check{\mu}}_{\check{\beta}}]$. Note that the whole expression is equal to zero by the definition that vielbeins are holomorphic, because this implies that $\hat{w}^{\mu} = e^{\mu}_{\alpha}(z)dz^{\alpha}$. Now both the real and imaginary part of the expression vanish, yielding

$$e^{\mu}_{\beta} = e^{\check{\mu}}_{\check{\beta}} \qquad e^{\mu}_{\check{\beta}} = -e^{\check{\mu}}_{\beta}.$$

The $\mu\lambda$ component then becomes

$$e^{\mu}_{k} e^{k}_{\lambda} = \delta^{\mu}_{\lambda} = e^{\mu}_{\kappa} e^{\kappa}_{\lambda} + e^{\mu}_{\check{\kappa}} e^{\check{\kappa}}_{\lambda} = e_{R}{}^{\mu}_{a} e_{R}{}^{a}_{\lambda} + e_{I}{}^{\check{\mu}}_{a} e_{I}{}^{a}_{\check{\lambda}}. \quad (22)$$

We can see that $C^{\mu\bar{\epsilon}} C_{\bar{\epsilon}\nu} = g^{\mu\epsilon} g_{\epsilon\nu} + B^{\mu\epsilon} B_{\epsilon\nu}$, which is equal to the $\mu\nu$ component of $\boldsymbol{g}^{me}\boldsymbol{g}_{en}$, is indeed equal to δ^{μ}_{ν}, using the inverse relations of the real and imaginary parts of the vielbeins (22).

IV. FLAT SPACE

The Hermitian line element in flat space becomes

$$ds^2 = -(cdt)^2 + (d\vec{x})^2 - (dy^0)^2 + (d\vec{y})^2. \quad (23)$$

From now on we declare the y coordinate to be the energy-momentum coordinate by defining[5] $y^{\mu} \equiv p^{\mu} \frac{G_N}{c^3}$. The space-time-momentum-energy interval squared from the origin to a space-time-momentum-energy point $\boldsymbol{x} = (ct, \vec{x}, \frac{G_N}{c^4}E, \frac{G_N}{c^3}\vec{p})$ is given by

$$d^2(\mathbf{0}; \boldsymbol{x}) = -(ct)^2 + (\vec{x})^2 + \frac{G_N^2}{c^6}\left[(\vec{p})^2 - \left(\frac{E}{c}\right)^2\right], \quad (24)$$

where G_N^2/c^6 suppresses the momentum-energy part by a factor on the order of $10^{-72}s^2/kg^2$, as it should do, since we do not observe any momentum-energy contributions at low energies.

The group of transformations, which leaves the Hermitian metric (12) invariant, is the $U(1,3)$ group; the elements U of the $U(1,3)$ group satisfy by definition the relation $U^{\dagger}\eta U = \eta$. When considering only one space and one momentum dimension, instead of three of each, we can represent the elements of the $SU(1,1)$ group [10], operating on the space-time-momentum-energy vector (t, x, p, E), by

$$\Gamma(v, f, f_0) = \gamma(v, f, f_0) \begin{pmatrix} 1 & \frac{v}{c^2} & \frac{G_N^2 f}{c^8} & -\frac{G_N^2 f_0}{c^9} \\ v & 1 & \frac{G_N^2 f_0}{c^7} & -\frac{G_N^2 f}{c^8} \\ f & -\frac{f_0}{c} & 1 & \frac{v}{c^2} \\ cf_0 & -f & v & 1 \end{pmatrix},$$

where $\gamma(v, f, f_0) = \left(1 - \frac{v^2}{c^2} - \frac{G_N^2 f^2}{c^8} + \frac{G_N^2 f_0^2}{c^8}\right)^{-\frac{1}{2}}$ and where $f = \dot{p}$ and $f_0 = \frac{\dot{E}}{c}$. We have neglected two space and two momentum components for convenience, since the transformations between the spacial components are simply rotations in space. An overall phase factor can be added later if one wants to consider the $U(1,1)$ group instead of the $SU(1,1)$ group. Note that the $SU(1,3)$ group reduces to its subgroup the Lorentz group, the $SO(1,1)$ group, for f_i and f_0 being zero. Thus, for inertial frames the theory clearly reduces to the laws of special relativity.

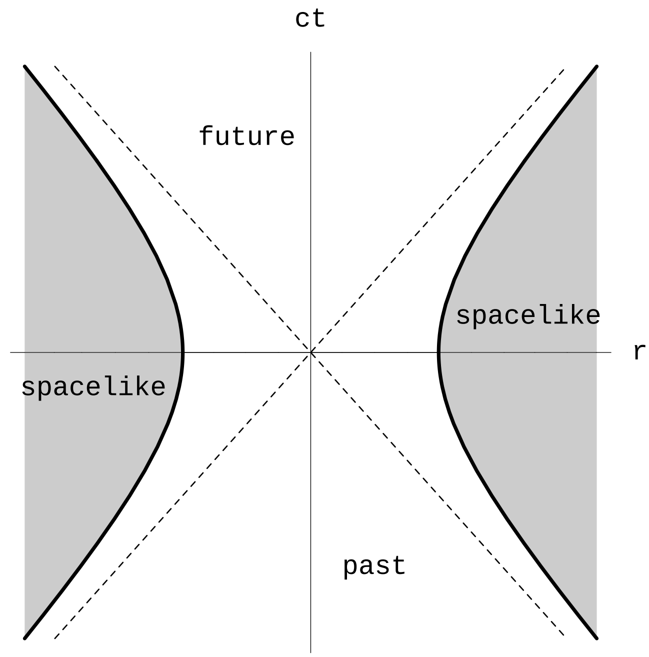

FIG. 1: A light cone, modified by non inertial coordinate transformations, is being portrayed on a space-time-momentum-energy diagram, separating the regions that are in causal contact with each other, from the regions that are not. There is a nonlocal causally related volume element at the origin.

If we set $ds^2 = 0$ and there is no momentum-energy contribution, we know that $\dot{\vec{x}} = c$. Demanding again that $ds^2 = 0$, but now for non vanishing momentum energy contributions however, the space-time-momentum-energy interval (24) becomes

$$-(ct)^2 + (\vec{x})^2 + \frac{G_N^2 p^2}{c^6} = 0, \quad (25)$$

where $p^2 = p^{\mu}p_{\mu}$ measures the energy-momentum in a space-time-momentum-energy hyper surface. Setting the space-time-momentum-energy interval to zero determines the causality boundary: the hypersurface specified by this condition determines the boundary of the causally related regions (for causally related events the '$=$' sign in (25) should be replaced by a '$\leq$').

Consider now the case when the momentum-energy contribution p^2 is negative. We can furthermore restrict the hypersurface by setting t to zero. Solving for $\vec{x}$ we obtain

$$r_{\max} = \frac{G_N\sqrt{-p^2}}{c^3},$$

[5] In principle one can pick any constant such that the units come out right. If one decides to construct the constant without introducing new constants, Newton's constant over the speed of light cubed is a unique choice up to factors of order unity. Since we are, up to this point, constructing a classical theory, there is no room for Planck's constant.

where $r_{\rm max} = \sqrt{\vec{x}^2(t=0)}$ is the maximal radius for a spatial volume element which is simultaneously causally related in a nonlocal manner. This can be seen from figure (1), which depicts this causality boundary. The existence of a momentum dependent maximal radius (often referred to as a minimal length scale) suggests that nature is inherently nonlocal for non-inertial observers. When p^2 is zero the standard light-cone is recovered. To get a feeling on how big this 'violation of causality' can be, let us consider a particle on the momentum-energy shell (corresponding to the traditional *on-shell* notion), in which case $p^2 = -m^2c^2$, such that $r_{\rm max} = G_N m/c^2 = r_{\rm Sch}/2$, where $r_{\rm Sch}$ denotes the conventional Schwarzschild radius. Recall that our flat-space analysis is based on the geodesic equation and its integral the line element, and hence completely neglects the self-gravity of (elementary) particles. When the self-gravity effects are included however, we expect that the above-discussed violation of causality gets hidden by the Schwarzschild radius created by the particles in consideration. It would be of interest to consider in detail how the causality analysis gets modified when the self gravity of particles is taken into account.

Alternatively, we can write $r_{\rm max} = G_N m/c^2 = (m/m_{\rm Pl})\ell_{\rm Pl}$, where we introduced the conventional Planck mass $m_{\rm Pl} = \sqrt{c\hbar/G_N} \simeq 2.18\times 10^{-8}$kg and the Planck length $\ell_{\rm Pl} = \sqrt{\hbar G_N/c^3} \simeq 1.616\times 10^{-35}$m, and where $\hbar$ denotes the Planck constant. Even though $r_{\rm max}$ is a purely classical quantity, when represented in terms of the Planck units, the Planck constant appears (which gets, of course, cancelled in the ratio $\ell_{\rm Pl}/m_{\rm Pl}$).

We can also specify our hypersurface differently by setting $\vec{x}$ to zero instead of t (this is in the case of positive momentum-energy, p^2). Solving for t we obtain

$$t_{\rm min} = \frac{G_N\sqrt{p^2}}{c^4},$$

where $t_{\rm min}$ is the minimal time for which an event is causally related to past space-time-momentum-energy events, as can be seen from the space-time-momentum-energy diagram, figure 2. This means that $t_{\rm min}$ is the minimal time for which an event can influence future events. Equivalently, $t_{\rm min}$ is the minimal time for which an event can be influenced by past events. In the light of the above discussion of $r_{\rm max}$, we see that the minimal time $t_{\rm min}$ shown in figure 2 exists only for particles moving off-shell, and hence these particles are strictly speaking not classical.

The definitions of space-time-momentum-energy past and future are natural generalizations of space-time past and future; the regions that are in causal contact are given again by events removed from the origin by space-time-momentum-energy intervals smaller or equal then zero, $ds^2 \le 0$. We have thus shown that in the limit of weak fields, Hermitian gravity exhibits simultaneously connected regions (maximal distance) for time like energy-momentum intervals and a breach of causality (minimal time) for space-like frame energy-momenta.

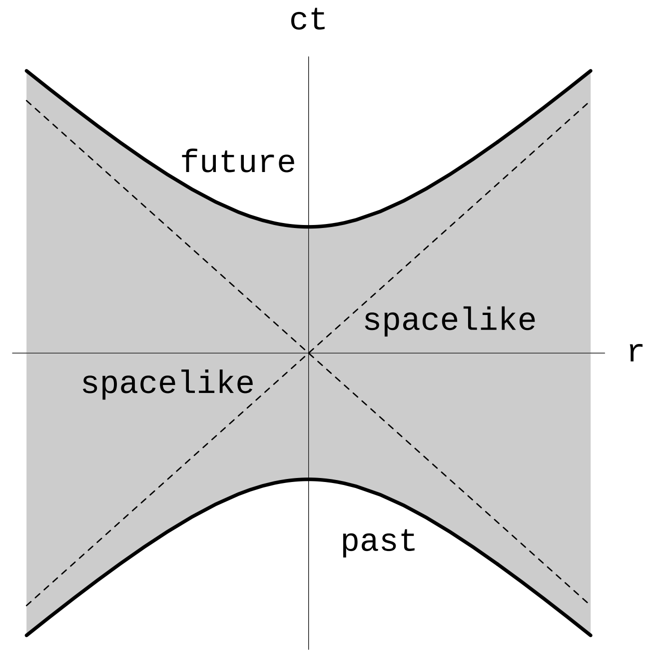

FIG. 2: A light cone, modified by non inertial coordinate transformations, is being portrayed on a space-time-momentum-energy diagram separating the regions that are in causal contact each other, from the regions that are not. There is a minimal time interval for events to be in causal contact.

For light-like frame energy-momenta, Hermitian gravity reproduces the standard light causality structure characterizing the weak field limit of general relativity.

We now calculate the phase velocity, using the space-time-momentum-energy line element (25)

$$v_{\rm phase} = \frac{\|\vec{r}\|}{t} = \sqrt{c^2 - \frac{G_N^2}{c^6}\frac{p^2}{t^2}}\,. \tag{26}$$

One can see that the phase velocity approaches the conventional speed of light c for large t. For times smaller than the minimal time the phase velocity becomes imaginary and damping will occur. The group velocity for massless particles ($ds/dt = 0$) becomes

$$v_g = \sqrt{c^2 - \frac{G_N^2}{c^6}f^2}, \tag{27}$$

where the four force squared is given by

$$f^2 = -\frac{1}{c^2}\left(\frac{dE}{dt}\right)^2 + \left(\frac{d\vec{p}}{dt}\right)^2. \tag{28}$$

The maximal group velocity approaches the speed of light for small four forces squared f^2. The maximal group velocity must be real in order to facilitate propagation. The reality requirement implies $f^2 \le c^4/G_N$. This can be seen from the space-time-momentum-energy line element (23) divided by $(dt)^2$. Upon solving (23) for the four force squared (28), we obtain $(G_N^2/c^6)f^2 \equiv$

$(G_N^2/c^6)\,(dp/dt)^2 = c^2 - v_g^2 + (ds/dt)^2 \le c^2 - v_g^2 \le c^2$, where the inequalities follow from the observations that $(ds/dt)^2 \le 0$ and $v_g^2 \ge 0$ (the reality condition on v_g in Eq. (27)). This then implies

$$f^2 \le f_{\max}^2 = \frac{c^8}{G_N^2}.$$

There is no lower bound on f^2, since $(ds/dt)^2 \le 0$ can at least in principle be arbitrarily large and negative. This means that there is also no upper bound on the maximal group velocity (27), as f^2 can in principle be very large and negative. While this may be true in principle, more realistically – in the flat space limit and in the absence of external forces – the flat space geodesic equation implies that $dz^\mu/d\tau = U^\mu = (\text{constant})^\mu$, from which we conclude that f^2 must be constant, such that the (maximal allowed) group velocity (27) acquires a *constant* correction in flat spaces and in the absence of external forces. This type of corrections can play an important role in strongly curved space-times however, where strong gravitational forces exist, which are expected to induce large changes in f^2, and thus possibly *superluminal* propagation. The change of causality structure discussed in this section deserves a deeper analysis, since it might have important consequences for the physics of structure formation in the early universe [11, 12].

V. THE EQUATIONS OF MOTION

In order to write down the equations of motion for this Hermitian theory of gravity, one needs to know what the connection coefficients are. There are already connection coefficients which are called Hermitian connection coefficients [7]. We will derive different connection coefficients later, but in order to appreciate these newly obtained connection coefficients, we will consider the old ones first.

A. The Known Connection Coefficients

One can derive the known Hermitian connection coefficients easily if one requires metric compatibility of the Hermitian metric and the fact that the holomorphic covariant derivative of an anti-holomorphic basis vector vanishes. The vanishing of he holomorphic covariant derivative of an anti-holomorphic basis vector is given by

$$\nabla_\mu \frac{\partial}{\partial z^{\bar\nu}} = 0 \qquad\qquad \nabla_{\bar\mu} \frac{\partial}{\partial z^{\nu}} = 0. \tag{29}$$

This implies that $\Gamma(\text{mixed indices}) = 0$, since the complex connection coefficients are usually defined as

$$\nabla_\mu \frac{\partial}{\partial z^{\bar\nu}} = \Gamma^{\epsilon}_{\mu\bar\nu} \frac{\partial}{\partial z^{\epsilon}} = 0.$$

If one then imposes metric compatibility on the Hermitian metric

$$\nabla_\rho C_{\bar\mu\nu} = \partial_\rho C_{\bar\mu\nu} - C_{\bar\mu\lambda}\Gamma^{\lambda}_{\rho\nu} = 0 \tag{30a}$$

$$\nabla_{\bar\rho} C_{\bar\mu\nu} = \partial_{\bar\rho} C_{\bar\mu\nu} - C_{\bar\lambda\nu}\Gamma^{\bar\lambda}_{\bar\rho\bar\mu} = 0 \tag{30b}$$

one can easily read off the Hermitian connection coefficients

$$\Gamma^{\lambda}_{\rho\nu} = C^{\bar\epsilon\lambda}\partial_\rho C_{\nu\bar\epsilon} \qquad\qquad \Gamma^{\bar\lambda}_{\bar\rho\bar\nu} = C^{\bar\lambda\epsilon}\partial_{\bar\rho} C_{\epsilon\bar\nu}. \tag{31}$$

If one looks carefully at the Hermitian metric compatibility equations (30) one can easily see that these equations imply vielbein compatibility; one can obtain the Hermitian connection coefficients by imposing vielbein compatibility in the following manner

$$\nabla_\mu e_\nu = 0 \quad \nabla_{\bar\mu} e_\nu = 0 \quad \nabla_\mu e_{\bar\nu} = 0 \quad \nabla_{\bar\mu} e_{\bar\nu} = 0. \tag{32}$$

Theories of Hermitian gravity, satisfying vielbein compatibility, have been proposed [13]. We shall see below that in such theories the geodesic equation is not obtained via an action principle. In an attempt to fix this problem we will weaken this vielbein compatibility condition and obtain different connection coefficients.

Note that the two independent components of the Riemann tensor,

$$\begin{aligned} R^{\kappa}_{\lambda\bar\mu\nu} &= \partial_{\bar\mu}\Gamma^{\kappa}_{\nu\lambda} = \partial_{\bar\mu}\left(C^{\bar\epsilon\kappa}\partial_\nu C_{\lambda\bar\epsilon}\right) \\ R^{\bar\kappa}_{\bar\lambda\mu\bar\nu} &= \partial_{\mu}\Gamma^{\bar\kappa}_{\bar\nu\bar\lambda} = \partial_{\mu}\left(C^{\bar\kappa\epsilon}\partial_{\bar\nu} C_{\epsilon\bar\lambda}\right)\,, \end{aligned} \tag{33}$$

contain only first order derivatives, when assuming the Hermitian metric to be a product of a holomorphic and anti-holomorphic vielbein $C_{\bar\mu\nu} = e^{a}_{\bar\mu}(\bar z^\gamma)\eta_{ab}e^{b}_{\nu}(z^\gamma)$. This implies that if one would attempt to write a complex equation of motion, analogues to Einstein's equations, one would obtain a first order differential equation. This means that the space-time-momentum-energy curvature for this theory is non-dynamical.

B. The Hermitian Geodesic Equations

The easiest way to derive the connection coefficients for Hermitian gravity is through varying the Hermitian line element. One then obtains the Hermitian geodesic equations from which one can read off the connection coefficients. Varying the Hermitian line element is a very easy exercise, when considering its eight dimensional form (8). One then obtains the eight dimensional complex geodesic equations

$$\ddot{\boldsymbol{z}}^r + \boldsymbol{\Gamma}^{r}_{mn}\dot{\boldsymbol{z}}^m\dot{\boldsymbol{z}}^n = \boldsymbol{0}, \tag{34}$$

where the complex connection coefficients are given by

$$\boldsymbol{\Gamma}^{r}_{mn} = \frac{1}{2}\boldsymbol{C}^{re}(\boldsymbol{\partial}_m\boldsymbol{C}_{en} + \boldsymbol{\partial}_n\boldsymbol{C}_{me} - \boldsymbol{\partial}_e\boldsymbol{C}_{mn}). \tag{35}$$

The Hermitian metric is defined such that the $\mu\bar{\nu}$ component of $\boldsymbol{C}_{mn}$ is $C_{\mu\bar{\nu}}$, with vanishing unmixed components. The Hermitian geodesic equations are then given by

$$\begin{aligned}\ddot{z}^{\rho}+\Gamma^{\rho}_{\mu\nu}\dot{z}^{\mu}\dot{z}^{\nu}+\Gamma^{\rho}_{\bar{\mu}\nu}\dot{z}^{\bar{\mu}}\dot{z}^{\nu}+\Gamma^{\rho}_{\mu\bar{\nu}}\dot{z}^{\mu}\dot{z}^{\bar{\nu}}+\Gamma^{\rho}_{\bar{\mu}\bar{\nu}}\dot{z}^{\bar{\mu}}\dot{z}^{\bar{\nu}} &= 0\\ \ddot{z}^{\bar{\rho}}+\Gamma^{\bar{\rho}}_{\mu\nu}\dot{z}^{\mu}\dot{z}^{\nu}+\Gamma^{\bar{\rho}}_{\bar{\mu}\nu}\dot{z}^{\bar{\mu}}\dot{z}^{\nu}+\Gamma^{\bar{\rho}}_{\mu\bar{\nu}}\dot{z}^{\mu}\dot{z}^{\bar{\nu}}+\Gamma^{\bar{\rho}}_{\bar{\mu}\bar{\nu}}\dot{z}^{\bar{\mu}}\dot{z}^{\bar{\nu}} &= 0,\end{aligned}$$

where the connection coefficients are

$$\begin{aligned}\Gamma^{\rho}_{\mu\nu} &= \frac{1}{2}C^{\bar{\lambda}\rho}(\partial_{\mu}C_{\nu\bar{\lambda}}+\partial_{\nu}C_{\mu\bar{\lambda}}) \qquad (36)\\ \Gamma^{\rho}_{\bar{\mu}\nu} &= \frac{1}{2}C^{\bar{\lambda}\rho}(\partial_{\bar{\mu}}C_{\nu\bar{\lambda}}-\partial_{\bar{\lambda}}C_{\nu\bar{\mu}})\\ \Gamma^{\rho}_{\mu\bar{\nu}} &= \frac{1}{2}C^{\bar{\lambda}\rho}(\partial_{\bar{\nu}}C_{\mu\bar{\lambda}}-\partial_{\bar{\lambda}}C_{\mu\bar{\nu}})\\ \Gamma^{\bar{\rho}}_{\bar{\mu}\bar{\nu}} &= \frac{1}{2}C^{\bar{\rho}\lambda}(\partial_{\bar{\mu}}C_{\lambda\bar{\nu}}+\partial_{\bar{\nu}}C_{\lambda\bar{\mu}})\\ \Gamma^{\bar{\rho}}_{\mu\bar{\nu}} &= \frac{1}{2}C^{\bar{\rho}\lambda}(\partial_{\mu}C_{\lambda\bar{\nu}}-\partial_{\lambda}C_{\mu\bar{\nu}})\\ \Gamma^{\bar{\rho}}_{\bar{\mu}\nu} &= \frac{1}{2}C^{\bar{\rho}\lambda}(\partial_{\nu}C_{\lambda\bar{\mu}}-\partial_{\lambda}C_{\nu\bar{\mu}})\\ \Gamma^{\rho}_{\bar{\mu}\bar{\nu}} &= 0\,, \qquad \Gamma^{\bar{\rho}}_{\mu\nu}=0.\end{aligned}$$

These connection coefficients are Hermitian in the following sense, $\overline{\Gamma^{\rho}_{\mu\bar{\nu}}}=\Gamma^{\bar{\rho}}_{\bar{\mu}\nu}{}^{T}$ and $\overline{\Gamma^{\rho}_{\mu\nu}}=\Gamma^{\bar{\rho}}_{\bar{\mu}\bar{\nu}}{}^{T}$. In eight dimensional form the connection coefficients are symmetric, $(\boldsymbol{\Gamma}^{r}_{mn})=(\boldsymbol{\Gamma}^{r}_{nm})$, just as the Levi-Civita symbols in general relativity. The wisdom of the eight dimensional notation becomes apparent now; for any known equation of general relativity one can sum over the barred and unbarred indices and plug in the just derived connection coefficients. If one plugs in the connection coefficients (36) into the Hermitian geodesic equations one obtains

$$\begin{aligned}\ddot{z}^{\rho}+\frac{1}{2}C^{\rho\bar{\lambda}}(\partial_{\mu}C_{\nu\bar{\lambda}}+\partial_{\nu}C_{\bar{\lambda}\mu})\dot{z}^{\mu}\dot{z}^{\nu}+&\\ C^{\rho\bar{\lambda}}(\partial_{\bar{\nu}}C_{\mu\bar{\lambda}}-\partial_{\bar{\lambda}}C_{\bar{\nu}\mu})\dot{z}^{\mu}\dot{z}^{\bar{\nu}}&=0\end{aligned}$$

and its Hermitian conjugate. One obtains precisely this result, when varying the particle action (with a mass m),

$$S=-m\int ds$$

with respect to the real one dimensional parameter proper time τ, where ds represents the Hermitian line element (12). Note that the known Hermitian connection coefficients (31) cannot be derived from any variation principle and therefore, in that sense, cannot have any physical meaning.

From general relativity we know that the metric transforms as

$$\boldsymbol{C}_{mn}\rightarrow\boldsymbol{C}_{mn}+\boldsymbol{\nabla}_{m}\boldsymbol{\Lambda}_{n}+\boldsymbol{\nabla}_{n}\boldsymbol{\Lambda}_{m}.$$

One can check that for example the $\mu\bar{\nu}$ component of this equation with the connection coefficients (36) plugged in corresponds indeed to the transformation of the Hermitian metric $C_{\mu\bar{\nu}}$, where $z^{\mu}\rightarrow z^{\mu}+\Lambda^{\mu}$ and $C_{\mu\bar{\nu}}\rightarrow C_{\mu\bar{\nu}}+\Lambda^{\epsilon}\partial_{\epsilon}C_{\mu\bar{\nu}}+\Lambda^{\bar{\epsilon}}\partial_{\bar{\epsilon}}C_{\mu\bar{\nu}}$. This should strengthen our belief in the connection coefficients (36). Note that via this procedure we can only obtain the coefficients with mixed indices. The unmixed coefficients can be obtained from the variation of the particle action with respect to the real variable proper time, because this variable breaks homomorphy in the sense that it depends on both z^{μ} and $z^{\bar{\mu}}$.

Finally, since we can simply rotate any tensorial equation from $z^{\mu}, z^{\bar{\mu}}$ space to $x^{\mu}, y^{\check{\mu}}$ space, it is useful to have expressions for the rotated connection coefficients, such that we can just plug these rotated coefficients into the rotated equations. Although the connection transforms by definition as a connection, it clearly transforms as a $(1,2)$ tensor under these rotations. This is the case, because these rotations are just constant transformations. Hence we can just replace the complex metric, $\boldsymbol{C}_{mn}$, by the rotated metric, $\boldsymbol{g}_{mn}$, in the expression of the eight dimensional connection (35) [14]. We state two components of the rotated connection coefficients

$$\begin{aligned}\Gamma^{\rho}_{\mu\nu} &= \frac{1}{2}g^{\rho\epsilon}(\partial_{\mu}g_{\epsilon\nu}+\partial_{\nu}g_{\mu\epsilon}-\partial_{\epsilon}g_{\mu\nu})\\ &+\frac{1}{2}B^{\rho\check{\epsilon}}(\partial_{\mu}B_{\check{\epsilon}\nu}+\partial_{\nu}B_{\mu\check{\epsilon}}-\partial_{\check{\epsilon}}g_{\mu\nu})\\ \Gamma^{\rho}_{\check{\mu}\nu} &= \frac{1}{2}g^{\rho\epsilon}(\partial_{\check{\mu}}g_{\epsilon\nu}+\partial_{\nu}B_{\check{\mu}\epsilon}-\partial_{\epsilon}B_{\check{\mu}\nu})\\ &+\frac{1}{2}B^{\rho\check{\epsilon}}(\partial_{\check{\mu}}B_{\check{\epsilon}\nu}+\partial_{\nu}g_{\check{\mu}\check{\epsilon}}-\partial_{\check{\epsilon}}B_{\check{\mu}\nu}).\end{aligned}$$

C. Torsion and Curvature

The Hermitian torsion[6] tensor T and the Riemann tensor R are defined as

$$\begin{aligned}T(Z,W) &= \nabla_{Z}W-\nabla_{W}Z-[Z,W]\\ R(Z,W)V &= \nabla_{Z}\nabla_{W}V-\nabla_{W}\nabla_{Z}V-\nabla_{[Z,W]}.\end{aligned}$$

The covariant derivative acting on a basis vector yields

$$\boldsymbol{\nabla}_{m}\frac{\boldsymbol{\partial}}{\boldsymbol{\partial z}^{n}}=\boldsymbol{\Gamma}^{e}_{mn}\frac{\boldsymbol{\partial}}{\boldsymbol{\partial z}^{e}}.$$

Note that the $\mu\bar{\nu}$ component of the covariant derivative acting on the basis vector is this time non-vanishing, unlike the $\mu\bar{\nu}$ component of the covariant derivative acting on the basis vector (29), using the known coefficients (31). We can now write the expressions for the components of the Hermitian torsion tensor

$$\begin{aligned}\boldsymbol{T}^{l}{}_{mn} &= \langle\hat{\boldsymbol{e}}^{l},\boldsymbol{T}(\hat{\boldsymbol{e}}_{m},\hat{\boldsymbol{e}}_{n})\rangle=\langle\hat{\boldsymbol{e}}^{l},\boldsymbol{\nabla}_{m}\hat{\boldsymbol{e}}_{n}-\boldsymbol{\nabla}_{n}\hat{\boldsymbol{e}}_{m}\rangle\\ &= \langle\hat{\boldsymbol{e}}^{l},\boldsymbol{\Gamma}^{b}_{mn}\hat{\boldsymbol{e}}_{b}-\boldsymbol{\Gamma}^{b}_{nm}\hat{\boldsymbol{e}}_{b}\rangle=\boldsymbol{\Gamma}^{l}_{mn}-\boldsymbol{\Gamma}^{l}_{nm}=0\end{aligned}$$

[6] In this article we define dynamical torsion to be a second order differential equations constraining the anti-symmetric part of the metric (19). This has in principle nothing to do with the vanishing or nonvanishing of the torsion tensor.

and the Hermitian Riemann tensor

$$\boldsymbol{R}^s_{mln} = \boldsymbol{\partial}_l \boldsymbol{\Gamma}^s_{nm} - \boldsymbol{\partial}_n \boldsymbol{\Gamma}^s_{lm} + \boldsymbol{\Gamma}^s_{la}\boldsymbol{\Gamma}^a_{nm} - \boldsymbol{\Gamma}^s_{na}\boldsymbol{\Gamma}^a_{lm},$$

with the connection coefficients (36) plugged in. Hence the Hermitian Riemann tensor is Hermitian in the following sense $\overline{R_{\bar{\mu}\nu}} = R_{\mu\bar{\nu}}{}^T$. Therefore the Hermitian Ricci scalar is real, $\overline{R} = R$.

D. Action principle for Hermitian gravity

The action for Hermitian gravity can be formulated as,

$$S[\boldsymbol{C}, \psi_i] = S_{hg}[\boldsymbol{C}] + S_c[\boldsymbol{C}] + S_M[\boldsymbol{C}, \psi_i] \tag{37}$$

where the pure gravity action is the following generalization of the Hilbert-Einstein action,

$$S_{hg}[\boldsymbol{C}] = \frac{1}{16\pi G_N}\int d\boldsymbol{z}^8 \sqrt{\boldsymbol{C}}(\boldsymbol{R} - 2\Lambda)\,,$$

where $\boldsymbol{R} = \boldsymbol{C}^{mn}\boldsymbol{R}_{mn}$ denotes the Ricci scalar, Λ cosmological constant and $\boldsymbol{C}^{mn}$ denotes the full complex metric tensor (7). For the reasons explained below, we impose the reciprocity symmetry only at the level of equations of motion (on-shell), which at the level of the action can be realized by a constraint. This of course means that physical quantities still respect the reciprocity symmetry. There is no unique way of imposing the reciprocity symmetry on the metric tensor. One reasonable choice is the following 'particle' action,

$$S_c[\boldsymbol{C}] = -M\int\Big[\lambda_1(C_{\mu\nu}dz^\mu dz^\nu \;+\; C_{\bar{\mu}\bar{\nu}}dz^{\bar{\mu}}dz^{\bar{\nu}}) \tag{38}$$
$$+\,\lambda_2(C_{\mu\bar{\nu}}dz^\mu dz^{\bar{\nu}} \;+\; C_{\bar{\mu}\nu}dz^{\bar{\mu}}dz^{\nu})\Big]^{1/2},$$

where M is a (mass) parameter and λ_1, λ_2 are Lagrange multipliers which break the symmetry between the holomorphic and Hermitian components of the complex metric (7). Taking, for example, $\lambda_1 = \lambda$ and $\lambda_2 = 1$, imposes $C_{\mu\nu} = 0 = C_{\bar{\mu}\bar{\nu}}$ and thus on-shell Hermiticity of the metric. Conversely, when $\lambda_1 = 1$ and $\lambda_2 = \lambda$ imposes $C_{\mu\bar{\nu}} = 0 = C_{\bar{\mu}\nu}$, implying on-shell holomorphy of the metric tensor. The constraint action (38) does not break holomorphy of the full theory (37) realized at the level of tetrads.

Just like the gravitational action, which obeys holomorphy at the level of vielbeins, we shall require that the matter action in (37) consists of holomorphic matter fields. Namely, holomorphy reduces the large number of degrees of freedom of the full *eight* dimensional theory to an acceptable number of degrees of freedom of an effectively *four* dimensional theory, as observationally required. For simplicity here we consider a matter action for scalar fields, which we use extensively below when we study cosmology. We consider two holomorphic scalar fields ϕ and ψ, one with Hermitian and one with holomorphic kinetic term, with the action:

$$S_M[\phi,\psi] = \int d^8\boldsymbol{z}\sqrt{\boldsymbol{C}}\,\mathcal{L}, \tag{39}$$

where the lagrangian density is given by

$$\mathcal{L} = -\frac{\alpha}{2}\boldsymbol{C}^{mn}(\partial_m\Phi)^\dagger\cdot\partial_n\Phi - \frac{\beta}{2}\boldsymbol{C}^{mn}(\partial_m\Psi)^T\cdot\partial_n\Psi - V$$

where

$$\Phi = \begin{pmatrix}\phi\\ \bar{\phi}\end{pmatrix}, \qquad \Psi = \begin{pmatrix}\psi\\ \bar{\psi}\end{pmatrix}, \tag{40}$$

where α and β are constants and where $V = V(\Phi,\Psi)$ is a potential. Note that $\Phi^\dagger\cdot\Phi = 2\phi\bar{\phi}$ and $\Psi^T\cdot\Psi = \psi^2 + \bar{\psi}^2$. The constants α and β can be absorbed in the fields ϕ and ψ by the appropriate field redefinitions, except for the sign of α, which is an invariant and thus can have physical relevance. For simplicity, we have assumed in Eq. (39) that the scalar fields do not couple to the Ricci scalar.

Varying the action (37) results in the Hermitian Einstein-Hilbert equations of motion

$$\begin{aligned}\boldsymbol{G}_{mn} + \Lambda\boldsymbol{C}_{mn} &= 8\pi G_N\boldsymbol{T}_{mn}\\ C_{\mu\nu} &= 0 = C_{\bar{\mu}\bar{\nu}}\,,\end{aligned} \tag{41}$$

where the second line equation is obtained by choosing $\lambda_1 = \lambda$, $\lambda_2 = 1$ and varying the action (38) with respect to λ. As usual the following definitions hold for the Einstein tensor $\boldsymbol{G}_{mn}$ and the stress energy tensor $\boldsymbol{T}_{mn}$:

$$\begin{aligned}\boldsymbol{G}_{mn} &= \boldsymbol{R}_{mn} - \frac{1}{2}\boldsymbol{C}_{mn}\boldsymbol{R}\,, \qquad \boldsymbol{R} = \boldsymbol{C}^{mn}\boldsymbol{R}_{mn}\\ \boldsymbol{T}_{mn} &= -\frac{2}{\sqrt{\boldsymbol{C}}}\frac{\delta S_M}{\delta\boldsymbol{C}^{mn}}\,.\end{aligned} \tag{42}$$

This formulation of the theory guarantees the (contracted) Bianchi identity, which in the eight dimensional form reads,

$$\boldsymbol{\nabla}^m\boldsymbol{G}_{mn} = 0.$$

The proof is analogous to that in general relativity. As a consequence, the stress energy must be covariantly conserved, $\boldsymbol{\nabla}^m\boldsymbol{T}_{mn} = 0$, just as desired. Note that imposing the reciprocity symmetry on the action (37) (off-shell) would result in an over-constrained on-shell dynamics which would fail to satisfy the Bianchi identity (43). We consider that as unacceptable, since that would imply nonconservation of the stress energy tensor, implying that energy would leak from our four dimensional spacetime hypersurface into the energy-momentum directions.

The stress energy tensor corresponding to the scalar field action (40) is just,

$$\boldsymbol{T}_{mn} = \alpha(\partial_m\Phi)^\dagger\cdot\partial_n\Phi + \beta(\partial_m\Psi)^T\cdot\partial_n\Psi + \boldsymbol{C}_{mn}\mathcal{L}\,, \tag{43}$$

where we used $\delta\sqrt{\boldsymbol{C}} = -\frac{1}{2}\sqrt{\boldsymbol{C}}\boldsymbol{C}_{mn}\delta\boldsymbol{C}^{mn}$.

When written in the four dimensional notation, Eqs. (41) reduce to,

$$G_{\mu\nu} = R_{\mu\nu} \ = \ 8\pi G_N T_{\mu\nu} \tag{44a}$$

$$G_{\mu\bar{\nu}} + C_{\mu\bar{\nu}}\Lambda \ = \ 8\pi G_N T_{\mu\bar{\nu}} \tag{44b}$$

plus the corresponding Hermitian conjugate equations, where $C_{\mu\nu} = 0$ and $G_{\mu\bar{\nu}} = R_{\mu\bar{\nu}} - \frac{1}{2}C_{\mu\bar{\nu}}R$. We also have, $R_{\mu\nu} = R^{\alpha}_{\ \mu\alpha\nu} + R^{\bar{\alpha}}_{\ \mu\bar{\alpha}\nu}$ and $R_{\mu\bar{\nu}} = R^{\alpha}_{\ \mu\alpha\bar{\nu}} + R^{\bar{\alpha}}_{\ \mu\bar{\alpha}\bar{\nu}}$. Note that the holomorphic equation (44a) does not admit a cosmological term Λ. Indeed, Λ is removed from (44a) by the on-shell reciprocity symmetry.

E. Metric Compatibility

When working in the first order formalism [14], in addition to Eqs. (41) one also obtains the metric compatibility equations,

$$\boldsymbol{\nabla}_m \boldsymbol{C}_{nr} = \boldsymbol{0}. \tag{45}$$

We list two components in four dimensional notation

$$\nabla_\rho C_{\mu\bar{\nu}} = 0\,, \qquad \nabla_\rho C_{\mu\nu} = 0. \tag{46}$$

One can check that the connection coefficients (36) are consistent with the metric compatibility condition (46): since $C_{\mu\nu} = 0$ and $\Gamma^{\bar{\rho}}_{\mu\nu} = 0$, we obtain that $\nabla_\rho C_{\mu\nu} = 0$. The mixed components[7] of the metric compatibility condition

$$\nabla_\rho C_{\mu\bar{\nu}} = \partial_\rho C_{\mu\bar{\nu}} - \Gamma^{\epsilon}_{\rho\mu} C_{\epsilon\bar{\nu}} - \Gamma^{\bar{\epsilon}}_{\rho\bar{\nu}} C_{\bar{\epsilon}\mu}$$

vanish as well

$$\partial_\rho C_{\mu\bar{\nu}} - \frac{1}{2}(\partial_\rho C_{\mu\bar{\nu}} + \partial_\mu C_{\rho\bar{\nu}}) - \frac{1}{2}(\partial_\rho C_{\mu\bar{\nu}} - \partial_\mu C_{\rho\bar{\nu}}) = 0.$$

On the other hand, writing the Hermitian metric component in terms of vielbeins and using the Leibnitz rule for the covariant derivative, one can show that the connection coefficients (36) do not imply the vielbein compatibility (32) discussed above. Indeed, writing Eq. (46) as

$$\nabla_\rho(e_\mu)e_{\bar{\nu}} + e_\mu \nabla_\rho e_{\bar{\nu}} = 0,$$

implies that $\nabla_\rho e_{\bar{\nu}} = -e_{\bar{\epsilon}}\Gamma^{\bar{\epsilon}}_{\rho\bar{\nu}} - e_\epsilon \Gamma^{\epsilon}_{\rho\bar{\nu}} \neq 0$ is not zero for instance.

[7] In order to show that $\nabla_\rho C_{\bar{\mu}\nu} = 0$, one needs to realize that $C^{\mu\bar{\epsilon}}C_{\nu\bar{\epsilon}} \equiv \beta^{\mu}_{\nu} \neq \delta^{\mu}_{\nu} = C^{\bar{\epsilon}\mu}C_{\nu\bar{\epsilon}}$, since the Hermitian metric is nonsymmetric (it is Hermitian). The action of β on the metric, $\beta^{\alpha}_{\nu}C_{\bar{\mu}\alpha} = C_{\nu\bar{\mu}}$, can be derived by inserting the identity in the Hermitian line element (12).

VI. COUNTING DEGREES OF FREEDOM

In order to get a glimpse of the general structure of Hermitian gravity, it might be educational to determine some properties of relevant objects, which are part of the theory. If we would like to know for instance, if we can always go to a freely falling frame, we can begin with counting the degrees of freedom of an arbitrary coordinate transformation of the metric tensor, in order to see if there are enough coordinate degrees of freedom in order to do so. If we then Taylor expand both sides of the coordinate transformation of the metric tensor

$$\tilde{C}_{\bar{\mu}\nu} = \frac{\partial z^{\bar{\alpha}}}{\partial \tilde{z}^{\bar{\mu}}}\frac{\partial z^{\beta}}{\partial \tilde{z}^{\nu}} C_{\bar{\alpha}\beta}, \tag{47}$$

we can collect terms of a specific order of the expansion of both sides of the equation and equate these terms. We are Taylor expanding both sides of the coordinate transformation of the Hermitian metric (48) around a point p on the (smooth) manifold.

When considering the zeroth order terms of the expansion,

$$\tilde{C}_{\bar{\mu}\nu}|_p = \frac{\partial z^{\bar{\alpha}}}{\partial \tilde{z}^{\bar{\mu}}}\frac{\partial z^{\beta}}{\partial \tilde{z}^{\nu}} C_{\bar{\alpha}\beta}\Big|_p, \tag{48}$$

we have 16 degrees of freedom at the left hand side of the equation, since the metric is Hermitian. The formula for the real degrees of freedom of a Hermitian matrix is d^2, where d is the complex dimension of the manifold, which is 4 in this case. On the right hand side of the equation, there are 32 real degrees of freedom to transform to the flat space metric (there are 32 degrees of freedom instead of 64, because the coordinate transformations are holomorphic, implying they satisfy the Cauchy-Riemann equations). Subtracting the two, we obtain $32 - 16 = 16$ degrees of freedom, which leave the flat space metric invariant. These 16 degrees of freedom are precisely the 16 degrees of freedom of the $U(1,3)$ group, which by definition leave the Hermitian flat space metric invariant.

Considering the following terms at first order of the expansion of the coordinate transformation (48)

$$\tilde{\boldsymbol{\partial}}_e \tilde{\boldsymbol{C}}_{mn}|_p + \cdots = \frac{\boldsymbol{\partial}^2 \boldsymbol{z}^a}{\boldsymbol{\partial}\tilde{\boldsymbol{z}}^e \boldsymbol{\partial}\tilde{\boldsymbol{z}}^m}\frac{\boldsymbol{\partial}\boldsymbol{z}^b}{\boldsymbol{\partial}\tilde{\boldsymbol{z}}^n}\boldsymbol{C}_{ab}\Big|_p + \cdots\,,$$

we count 64 real degrees of freedom on the left hand side and 80 on the right. We obtain the 64 real dimension in the following manner. The complex number of degrees of freedom for a Hermitian matrix is $\frac{1}{2}d^2$, where d is again the complex dimension of the manifold, which is, as said before, 4 in this case. The complex dimension of the partial derivative is d. When multiplying these numbers we obtain 32 complex degrees of freedom, which is equivalent to 64 real degrees of freedom. The 80 degrees of freedom from the first factor of the term on the right hand side are obtained as follows. The numerator has d complex degrees of freedom. The denominator has $\frac{1}{2}d(d+1)$ complex degrees of freedom, which is just the formula of a

symmetric matrix, since partial derivatives commute. By multiplying these numbers together we obtain $4\cdot 10 = 40$, complex degrees of freedom, which is equivalent to 80 real degrees of freedom. When subtracting these numbers we obtain $80-64=16$ real degrees of freedom. This means that we have 16 degrees of freedom too many in order to transform to the free falling frame.

Finally, considering the following two terms at second order of the expansion of the coordinate transformation of the Hermitian metric (48)

$$\tilde{\boldsymbol{\partial}}_e\tilde{\boldsymbol{\partial}}_f\tilde{\boldsymbol{C}}_{mn}|_p+\cdots=\frac{\boldsymbol{\partial}^3\boldsymbol{z}^a}{\boldsymbol{\partial}\tilde{\boldsymbol{z}}^e\boldsymbol{\partial}\tilde{\boldsymbol{z}}^f\boldsymbol{\partial}\tilde{\boldsymbol{z}}^m}\frac{\boldsymbol{\partial}\boldsymbol{z}^b}{\boldsymbol{\partial}\tilde{\boldsymbol{z}}^n}\boldsymbol{C}_{ab}\Big|_p+\cdots,$$

we have 160 real degrees of freedom at the left hand side and 160 on the right. We obtain the 160 real degrees of freedom on the left hand side as follows. The complex degrees of freedom of the partial derivatives is $\frac{1}{2}d(d+1)$ and the complex degrees of freedom of the metric is again $\frac{1}{2}d^2$. Multiplying these numbers together we obtain $10*8=80$ complex degrees of freedom, which is equivalent to 160 real degrees of freedom. The 160 real degrees of freedom on the right hand side are obtained as follows. The numerator has again dimension d. The denominator has dimension $\frac{1}{3!}d(d+1)(d+2)$. Multiplying these numbers together we obtain $4*20=80$ complex degrees of freedom, which is again equivalent to 160 real degrees of freedom. When subtracting these two numbers we obtain $160-160=0$ degrees of freedom. This means that we have precisely enough degrees of freedom in order to obtain flat space at second order of the expansion. Hence there is no space-time-momentum-energy curvature in the theory of Hermitian gravity. This might appear as problematic since general relativity does contain space-time curvature, which we cannot get rid off by coordinate transformations. We shall see below that in the limit of projecting space-time-momentum-energy onto space-time we will obtain space-time curvature as an artifact of the limiting procedure.

Hermitian gravity is dynamical in the sense that there are second order derivatives, acting upon the dynamical variable, the vielbein or the metric. Consider the following independent components of the Hermitian Riemann tensor,

$$R^{\kappa}_{\lambda\mu\nu}=C^{\bar{\epsilon}\kappa}\partial_\lambda\partial_{[\mu}C_{\nu]\bar{\epsilon}}+\text{first order derivatives},\quad (49)$$

and its Hermitian conjugate. Unlike the components of the Riemann tensor (33), constructed from the known complex connection coefficients (31), we do have non-vanishing components of the Riemann tensor, which do contain second order derivatives. These components *do* enter the Hermitian Einstein equations (44b), although $C_{\mu\nu}=0$. These components of the Hermitian Einstein tensor act as constraints, such that we remain on the hypersurface, which specified by the reciprocity transformation.

VII. THE LIMIT TO GENERAL RELATIVITY

The limit of Hermitian gravity to the theory of general relativity is based on the assumption that the y coordinate and its corresponding vielbein are small. When expanding these theories in powers of y and its corresponding vielbein, we would hope to obtain the theory of general relativity at zeroth order of the expansion and meaningful corrections to the theory at linear order. We will see that this is not the case, since we will obtain corrections to general relativity at zeroth order. The easiest way to obtain the limit to general relativity is to expand the real and the imaginary parts of the vielbein in terms of the y coordinate in order to collect the terms in orders of the y coordinate and its corresponding vielbein, $e_{I\tilde{\mu}}$, yielding

$$\begin{aligned} e_{R\mu}(x,y) &= e_{R\mu}(x)-y^\lambda\partial_\lambda e_{I\tilde{\mu}}+O(y^2) \quad (50a)\\ &= e_{R\mu}(x)+O'(y^2) \end{aligned}$$

and

$$e_{I\tilde{\mu}}(x,y)=e_{I\tilde{\mu}}(x)+y^\lambda\partial_\lambda e_{R\mu}(x)+O(y^2).\quad (50b)$$

These expansions contain sufficient information in order to obtain the limit to general relativity. We will, however, also expand the rotated metric components and a component of the rotated connection coefficients. Using the expansions of the real and imaginary parts of the vielbeins (50), the rotated metric components up to second order of the y coordinate and its corresponding vielbein are

$$g_{\mu\nu}(x,y)=g_{\mu\nu}(x)+O(y^2),$$

$$g_{\mu\tilde{\nu}}(x,y)=g_{\mu\tilde{\nu}}(x)+y^\lambda\left(\partial_\lambda(e_\mu)e_\nu-e_\mu\partial_\lambda e_\nu\right)+O(y^2),$$

$$g_{\tilde{\mu}\nu}(x,y)=g_{\tilde{\mu}\nu}(x)+y^\lambda\left(e_\mu\partial_\lambda e_\nu-\partial_\lambda(e_\mu)e_\nu\right)+O(y^2),$$

and

$$g_{\tilde{\mu}\tilde{\nu}}(x,y)=g_{\tilde{\mu}\tilde{\nu}}(x)+O(y^2),$$

where $g_{\mu\nu}(x)=g_{\tilde{\mu}\tilde{\nu}}(x)=e_\mu e_\nu(x)$ and where $g_{\tilde{\mu}\nu}(x)$ and $g_{\mu\tilde{\nu}}(x)$ can be just read off the expression of the rotated Hermitian metric in terms of vielbeins. Using again the expansions of the real and imaginary parts of the vielbeins (50), the $\Gamma^\rho_{\mu\nu}(x,y)$ component of the connection coefficients (36) up to second order of the y coordinate and its corresponding vielbein is

$$\Gamma^\rho_{\mu\nu}(x,y)=\Gamma^\rho_{\mu\nu}(x)+O(y^2),$$

where $\Gamma^\rho_{\mu\nu}(x)$ is just the ordinary Levi-Cività connection. With the expansions of the connection coefficients, we can now check if the theory of Hermitian gravity reduces to the theory of general relativity by plugging them into the rotated Hermitian geodesic equation, keeping only

terms of linear order in the y coordinate and its corresponding vielbein, yielding the ordinary geodesic equation

$$\ddot{x}^\rho + \Gamma^\rho_{\mu\nu}(x)\dot{x}^\mu\dot{x}^\nu + O(y^2) = 0$$

without any first order corrections present. Though the result is not spectacular at first sight, it should be pleasing that the theory of Hermitian gravity reduces to the well tested theory of general relativity, for the Hermitian geodesic equation. In order to see if the theory predicts any interesting new physics we have to collect terms up to second order, yielding

$$\begin{aligned}\ddot{x}^\rho &+ \Gamma^\rho_{\mu\nu}(x,y)\dot{x}^\mu\dot{x}^\nu + [\Gamma^\rho_{\check{\mu}\nu}(x,y) + \Gamma^\rho_{\nu\check{\mu}}(x,y)]\dot{y}^{\check{\mu}}\dot{x}^\nu \\ &+ \Gamma^\rho_{\check{\mu}\check{\nu}}(x,y)\dot{y}^{\check{\mu}}\dot{y}^{\check{\nu}} + O(y^3) = 0,\end{aligned}$$

where the connection coefficients $\Gamma^\rho_{\check{\mu}\nu}(x,y)$, $\Gamma^\rho_{\nu\check{\mu}}(x,y)$ and $\Gamma^\rho_{\check{\mu}\check{\nu}}$ are just the connection coefficients expanded up to linear order [14], but where the connection coefficient $\Gamma^\rho_{\mu\nu}(x,y)$ has to be expanded up to second order since the term $\dot{x}^\mu\dot{x}^\nu$, multiplying $\Gamma^\rho_{\mu\nu}(x,y)$, is of zeroth order in the y coordinate and its corresponding vielbein.

The Hermitian Einstein's equations get corrections to the Einstein's equations of general relativity at zeroth order. This can be seen when considering the rotated Hermitian Ricci tensor

$$R_{\mu\nu} = R^\lambda{}_{\mu\lambda\nu} + R^{\check{\lambda}}{}_{\mu\check{\lambda}\nu} = R^{GR}_{\mu\nu} + R_{\rm cor}{}^{\check{\lambda}}{}_{\mu\check{\lambda}\nu} + O(y^2),$$

where R^{GR} is the Ricci tensor according to general relativity and where $R_{\rm cor}$ are the terms of $R^{\check{\lambda}}{}_{\mu\check{\lambda}\nu}$ of zeroth order in the expansion in the $y^{\check{\mu}}$ coordinate. Similarly the rotated Hermitian Einstein tensor

$$\begin{aligned}G_{\mu\nu} &= G^{GR}_{\mu\nu} + R_{\rm cor}{}^{\check{\lambda}}{}_{\mu\check{\lambda}\nu} - \frac{1}{2}g_{\mu\nu}g^{\alpha\beta}R_{\rm cor}{}^{\check{\lambda}}{}_{\alpha\check{\lambda}\beta} \\ &-\frac{1}{2}g_{\mu\nu}g^{\check{\alpha}\check{\beta}}\left(R_{\rm cor}{}^{\lambda}{}_{\check{\alpha}\lambda\check{\beta}} + R_{\rm cor}{}^{\check{\lambda}}{}_{\check{\alpha}\check{\lambda}\check{\beta}}\right)] + O(y^2),\end{aligned}$$

gets corrections of zeroth order in the expansion in the $y^{\check{\mu}}$ coordinate. Hence, at this point one needs to look at the solutions of Hermitian gravity in order to see if the theory contradicts experiment or not.

VIII. HERMITIAN COSMOLOGY

In order to describe our Universe correctly, which is isotropic and homogeneous on large scales, our complex theory should permit solutions that possess the symmetries of isotropy and homogeneity and furthermore these solutions should correctly reduce to the Friedmann-Lemaître-Robertson-Walker (FLRW) cosmology at low energies. Since the vielbeins of our theory are holomorphic functions, we can demand spatial isotropy and homogeneity by implementing a scale factor that is a holomorphic function of the time-like coordinate only,

$$e^a_\mu = a(z^0)\delta^a_\mu \qquad e^a_{\bar{\mu}} = a(z^{\bar{0}})\delta^a_{\bar{\mu}}\,, \tag{51}$$

where the complex scale factor can be specified in terms of its real and imaginary parts

$$a(z^0) = a_R(z^0) + ia_I(z^0) \qquad \bar{a}(z^{\bar{0}}) = a_R(z^{\bar{0}}) - ia_I(z^{\bar{0}}).$$

Note that $z^0 = t + i\frac{G_N}{c^4}E$. The *Ansatz* (51) yields a cosmology with flat spatial sections, which suffices for our purpose.[8] With this *Ansatz* for the vielbein, the connection coefficients become holomorphic (or anti-holomorphic) functions. Consider for example the following two connection coefficients

$$\begin{aligned}\Gamma^\rho_{\mu\nu} &= \frac{1}{2}\frac{a'}{a}(\delta^\rho_\nu\delta^0_\mu + \delta^0_\nu\delta^\rho_\mu) \\ \Gamma^\rho_{\bar{\mu}\nu} &= \frac{1}{2}\frac{\bar{a}'}{\bar{a}}(\delta^{\bar{0}}_{\bar{\mu}}\delta^\rho_\nu - \eta^{\bar{0}\rho}\eta_{\bar{\mu}\nu}).\end{aligned} \tag{52}$$

These expressions for the connection coefficients can then be used to obtain the components of the Hermitian Ricci tensor. The expression for the mixed components of the Ricci tensor is then

$$R_{\bar{\mu}\nu} = \frac{\bar{a}'a'}{\bar{a}a}\left[\left(\frac{d-1}{2}\right)\delta^{\bar{0}}_{\bar{\mu}}\delta^0_\nu + (d-1)\eta_{\bar{\mu}\nu}\right],$$

where d is again the complex dimension of the manifold. Taking d to be four, we obtain the following expressions for the independent mixed components of the Hermitian Ricci tensor[9]

$$R_{\bar{0}0} = -\frac{3}{2}\frac{\bar{a}'a'}{\bar{a}a} \qquad R_{\bar{i}j} = 3\frac{\bar{a}'a'}{\bar{a}a}\eta_{\bar{i}j}$$

and the following expression for the Hermitian Ricci scalar

$$R = C^{\bar{\mu}\nu}R_{\bar{\mu}\nu} + C^{\mu\bar{\nu}}R_{\mu\bar{\nu}} = 21\frac{\bar{a}'a'}{(\bar{a}a)^2}.$$

The nonzero unmixed components are then

$$R_{00}(z^0) = \frac{9}{2}\left(\frac{a'}{a}\right)^2 - 3\left(\frac{a''}{a}\right)$$

and its complex conjugate. The independent components of the Hermitian Einstein tensor then become

$$\begin{aligned}G_{\bar{0}0} &= 9\frac{\bar{a}'a'}{\bar{a}a}\,, \qquad G_{\bar{i}j} = -\frac{15}{2}\frac{\bar{a}'a'}{\bar{a}a}\eta_{\bar{i}j} \\ G_{00} &= \frac{9}{2}\left(\frac{a'}{a}\right)^2 - 3\left(\frac{a''}{a}\right)\end{aligned} \tag{53}$$

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[8] To generalist the *Ansatz* (51) to space-times with a constant spatial curvature, one would have to replace $a(z^0)\delta^a_\mu$ in Eq. (51) by the corresponding vielbein whose spatial indices describe the geometry of a static 3-sphere (3-hyperboloid) for a space with positively (negatively) curved spatial sections. Thus for a space-time with positively curved spatial sections ($\kappa > 0$) we have, $e_\mu = a(z^0)[\delta^0_\mu + \delta^\chi_\mu + (1/\sqrt{\kappa})\sin(\sqrt{\kappa}\chi)\delta^\theta_\mu + (1/\sqrt{\kappa})\sin(\sqrt{\kappa}\chi)\sin(\theta)\delta^\varphi_\mu]$, where $\chi \in [0, \pi/\sqrt{\kappa}]$, $\theta \in [0,\pi]$ and $\varphi \in [0, 2\pi)$ are the spherical coordinates on S^3. For a space with a negative curvature ($\kappa < 0$) the tetrad e_μ is obtained from the tetrad of the closed universe with the replacement, $(1/\sqrt{\kappa})\sin(\sqrt{\kappa}\chi) \to (1/\sqrt{-\kappa})\sinh(\sqrt{-\kappa}\chi)$, where now $\chi \in [0,\infty)$.

[9] Here the Latin indices i and j take values $1, 2, 3$.

and their complex conjugates. If we now make use of the connection coefficients (52) we can easily check that the Einstein tensor of Hermitian gravity (53) obeys the Bianchi identity (43), implying that the Einstein tensor is divergenceless, as it should be. This represents a nontrivial check of the accuracy of our calculation.

Consider now the matter action. For definiteness (and simplicity) we shall consider the two scalar field action (39). Since in standard general relativity one can obtain any desired expansion $a = a(t)$ by appropriately choosing the scalar field potential, we expect that the action (39) does not pose any important restrictions to the Hermitian cosmology.

The underlying symmetries of a (flat) FLRW cosmology together with holomorphy then imply that the scalar fields are of the form, $\phi = \phi(z^0)$ and $\psi = \psi(z^0)$. With this observation we get that the nonvanishing components of the stress energy tensor (43) are,

$$\begin{aligned} T_{\mu\bar\nu} &= \alpha\delta^0_\mu\delta^{\bar 0}_{\bar\mu}\phi'\bar\phi' + \eta_{\mu\bar\nu}\big(\alpha\phi'\bar\phi' - a\bar a V\big) \\ T_{\mu\nu} &= \beta\delta^0_\mu\delta^0_\mu\psi'^2\,, \end{aligned} \tag{54}$$

plus the Hermitian conjugates. Here we used $\phi' = (\partial/\partial z^0)\phi$ and $\bar\phi' = (\partial/\partial z^0)\bar\phi$. The nonvanishing components in (54) are,

$$T_{0\bar 0} = a\bar a V \quad T_{i\bar j} = \delta_{i\bar j}\big(\alpha\phi'\bar\phi' - a\bar a V\big) \quad T_{00} = \beta\psi'^2. \tag{55}$$

When combined with Eqs. (53) these yield the following equations for Hermitian cosmology (44a–44b),

$$G_{00} \equiv \frac{9}{2}\left(\frac{a'}{a}\right)^2 - 3\left(\frac{a''}{a}\right) = 8\pi G_N\beta\psi'^2 \tag{56a}$$

$$G_{0\bar 0} + C_{0\bar 0}\Lambda \equiv 9\frac{\bar a' a'}{\bar a a} - a\bar a\Lambda = 8\pi G_N a\bar a V \tag{56b}$$

$$\begin{aligned} &\frac{1}{3}\delta^{i\bar j}\big(G_{i\bar j} + C_{i\bar j}\Lambda\big) \equiv -\frac{15}{2}\frac{\bar a' a'}{\bar a a} + a\bar a\Lambda \\ &= 8\pi G_N\big(\alpha\phi'\bar\phi' - a\bar a V\big)\,, \end{aligned} \tag{56c}$$

which together with the scalar field equations of motion,

$$\begin{aligned} -3\alpha\frac{\bar a'}{a\bar a^2}\phi' - \partial_{\bar\phi}V &= 0 \\ -3\beta\frac{\bar a'}{a\bar a^2}\psi' - \partial_\psi V &= 0 \end{aligned} \tag{57}$$

represent the closed system of equations of Hermitian cosmology with scalar fields. These equations are obtained by varying the matter action (39) with respect to $\bar\phi$ and ψ, respectively. The scalar equations of motion (57) can be can be also obtained from the covariant stress-energy conservation. Inspired by the form of stress-energy in FLRW spaces, $T_{\mu\nu} = a^2\delta_\mu^{\,0}\delta_\nu^{\,0}(\rho+p) + p g_{\mu\nu}$, the appropriate Hermitian gravity generalization is of the form,

$$\begin{aligned} T_{\mu\nu} &= (\rho_h + p_h)a^2\delta_\mu^{\,0}\delta_\nu^{\,0} \\ T_{\mu\bar\nu} &= a^2\delta_\mu^{\,0}\delta_{\bar\nu}^{\,\bar 0}(\rho+p) + pC_{\mu\bar\nu}\,. \end{aligned} \tag{58}$$

Comparing this with Eqs. (55) then implies

$$\rho = V \qquad p = \alpha\dot\phi\dot{\bar\phi} - V \qquad \rho_h + p_h = \beta\dot\psi^2 \tag{59}$$

where $(1/a)\phi' = \dot\phi$ and $(1/a)\psi' = \dot\psi$. While the pressure has a standard form, note that the kinetic term does not contribute to the energy density of Hermitian gravity. [10] The stress energy conservation $\nabla^m T_{mn} = 0$ then implies,

$$\dot\rho + 3\bar H(\rho_h + p_h) + 3H(\rho + p) = 0 \tag{60}$$

and the complex conjugate equation, where $\dot\rho = (1/a)\partial_0\rho$ and where $H = a'/a^2 = \dot a/a$ and $\bar H = \bar a'/\bar a^2 = \dot{\bar a}/\bar a$. This together with Eqs. (59) implies the scalar field equations (57), which checks the consistency of our formulation of Hermitian gravity with scalar fields.

Next, it is convenient to divide equation (56a) by a^2 and Eqs. (56b–56c) by $a\bar a$, respectively. Combining Eqs. (56b) and (56c) results in the constraint equations,

$$H\bar H = \frac{8\pi G_N}{9}\big(V + \lambda\big)\,, \qquad \lambda = \frac{\Lambda}{8\pi G_N} \tag{61a}$$

$$\dot\phi\dot{\bar\phi} = \frac{1}{6\alpha}\big(V + \lambda\big)\,. \tag{61b}$$

Equation (56a) can be recast as,

$$-\dot H - \frac{1}{2}H^2 = \frac{8\pi G_N}{3}\beta\dot\psi^2\,. \tag{61c}$$

With a help of Eq. (61a), the scalar equations (57) become,

$$\dot\phi = -\frac{3H}{8\pi G_N\alpha}\frac{\partial_{\bar\phi}V}{V+\lambda} \tag{61d}$$

$$\dot\psi = -\frac{3H}{8\pi G_N\beta}\frac{\partial_\psi V}{V+\lambda}\,. \tag{61e}$$

Equations (61a–61e) (and their hermitean conjugates) are the fundamental equations of hermitean cosmology. Note that there are 5 (2 real and 3 complex) equations for 3 complex quantities H, ϕ and ψ, so the system is overdetermined, and there is no guarantee that a solution exists.

We shall now show that a solution exists, and moreover we shall explicitly construct a class of solutions that gives rise to a power law expansion of the scale factor $a = a(z^0)$.

Firstly, Eq. (61a) implies that $\bar V = V$ and $\bar\lambda = \lambda$ are real. Secondly, Eq. (61d) and its complex conjugate imply that $\partial_{\bar\phi}\ln\big(V+\lambda\big)$ is a holomorphic functions of ϕ and that $\partial_\phi\ln\big(V+\lambda\big)$ is an antiholomorphic function of $\bar\phi$. Consequently the potential is determined to be of the form,

$$\ln(V+\lambda) = A_1\phi + \bar A_2\bar\phi + A_3\,, \tag{62}$$

[10] One would arrive at a more standard expression for the scalar energy density if one would replace $\rho \to (\rho - p)/2$ in Eq. (58).

where A_1, A_2 and A_3 are complex constants independent on ϕ (still possibly dependent on ψ and $\bar{\psi}$). The reality of V and λ then implies that $A_1 = A_2 \equiv \Omega$. Writing A_3 as $A_3 = \ln(W)$ we have,

$$V = -\lambda + W \exp\left[\Omega\big(\phi + \bar{\phi}\big)\right]. \tag{63}$$

Finally, since V is real, W and Ω must be real functions of ψ and $\bar{\psi}$. Holomorphy is such a powerful symmetry that - even though Eq. (63) represents the most general solution to Eqs. (61d) and its complex conjugate – the potentials V for ϕ is, up to two 'constants' W and Ω, completely fixed.

Now multiplying Eq. (61d) with its complex conjugate, making use of Eq. (61a) and inserting the resulting equation into (61b), we obtain,

$$\frac{4\pi G_N\alpha}{3} = \partial_{\bar{\phi}} \ln\big(V + \lambda\big)\partial_\phi \ln\big(V + \lambda\big) \equiv \Omega^2\,, \tag{64}$$

implying that there are two allowed values for Ω,

$$\Omega_\pm \equiv \pm\omega = \pm\sqrt{\frac{4\pi G_N\alpha}{3}}\,, \tag{65}$$

fixing thus Ω completely (up to a sign). (This sign ambiguity reflects the symmetry of the theory (39) under the transformation, $\phi \to -\phi$, $\bar{\phi} \to -\bar{\phi}$.) With this Eq. (61d) becomes

$$\dot{\phi} = \mp\frac{1}{2\omega}\frac{d}{dt}\ln(a)\,,$$

which is solved by

$$a = a_0 \mathrm{e}^{\mp 2\omega(\phi - \phi_0)}\,. \tag{66}$$

(ϕ_0 is unphysical as it can be absorbed in the definition of a_0.) This means that ϕ is not an independent field, but a constrained field which is just a reparametrization of the scale factor a. This is not surprising, given the fact that ϕ solves the constraint equations of Hermitian gravity.

The remaining equations to be solved are (61c) and (61e), and the remaining freedom in the potential is in $W = W(\psi, \bar{\psi})$. Similarly as above, we can see from Eq. (61e) that $\partial_\psi \ln(V + \lambda) = \partial_\psi \ln(W)$ must be a holomorphic function of ψ. We conclude that W must be a product of a holomorphic function of ψ and an anti-holomorphic function of $\bar{\psi}$ (and they must be mutually equal),

$$W = w(\psi)\bar{w}(\bar{\psi})\,. \tag{67}$$

With this observation, making use of (61e), Eq. (61c) can be recast as,

$$\epsilon \equiv -\frac{\dot{H}}{H^2} = \frac{1}{2} + \frac{3}{8\pi G_N\beta}\Big(\partial_\psi \ln(w(\psi))\Big)^2\,. \tag{68}$$

Let us now consider the case when $\epsilon = \text{const}$. In this case Eq. (68) can be easily solved for $w(\psi)$ in terms of ϵ,

$$w = w_0 \exp\left[\sqrt{\frac{8\pi G_N\beta}{3}\Big(\epsilon - \frac{1}{2}\Big)}\,\psi\right], \tag{69}$$

where w_0 is a field independent constant. The potential V that yields a power law expansion is therefore given by

$$\begin{aligned} V \;=\; & -\frac{\Lambda}{8\pi G_N} + V_0 \exp\big[\pm\omega(\phi + \bar{\phi})\big] \times \\ & \times\;\exp\left[\sqrt{\frac{8\pi G_N\beta}{3}}\bigg(\Big(\epsilon - \frac{1}{2}\Big)^{1/2}\psi + \Big(\bar{\epsilon} - \frac{1}{2}\Big)^{1/2}\bar{\psi}\bigg)\right], \end{aligned} \tag{70}$$

where $V_0 = w_0\bar{w}_0$ is a (real) constant and $\omega = \sqrt{4\pi G_N\alpha/3}$. Note that nonvanishing $\mathrm{Im}[\epsilon]$ breaks the charge-parity (CP) symmetry of the ψ field, as can be seen from the potential (70). However, CPT is conserved. We will discuss below how $\mathrm{Im}[\epsilon]$ breaks time reversal symmetry, T. Recall that a power law expansion means

$$\epsilon \equiv \frac{3}{2}(1 + w_f) = \text{const.}\,, \tag{71}$$

where $w_f = p/\rho$ is the (complex) equation of state parameter of a 'cosmological fluid' with a 'pressure' p and an 'energy density' ρ. Since $\epsilon = (d/dt)(1/H)$, a constant epsilon implies a power law expansion with,

$$H = \frac{1}{\epsilon z^0}\,, \qquad a(z^0) = a_0\left(\frac{z^0}{\zeta_0}\right)^{1/\epsilon}, \tag{72}$$

where a_0 and ζ_0 are (complex) constants. In standard cosmology, $\epsilon = 3/2$ ($\epsilon = 2$) correspond to matter (radiation) era, while $0 < \epsilon \ll 1$ corresponds to a slow roll inflation.

From Eqs. (61d–61e) and (70) we find,

$$\dot{\phi} = \mp\frac{H}{2\omega}\,, \qquad \dot{\psi} = -\frac{H}{2\omega}\sqrt{\frac{\alpha}{\beta}(2\epsilon - 1)}\,,$$

such that in a Universe expanding as a power law the two fields are not independent,

$$\psi - \psi_0 = \pm\sqrt{\frac{\alpha}{\beta}(2\epsilon - 1)}(\phi - \phi_0)\,. \tag{73}$$

When this is inserted into Eq. (66) we immediately get

$$a = a_0 \exp\left[-\sqrt{\frac{16\pi G_N\beta}{3(2\epsilon - 1)}}(\psi - \psi_0)\right]. \tag{74}$$

This means that – just like ϕ – in a power law expansion ψ corresponds to an ϵ-dependent reparametrization of the scale factor of the Universe. Note that $\epsilon = 1/2$ is a singular point of the relation (74). This is not unexpected, since from Eq. (61c) we know that $\epsilon = 1/2$ corresponds to the case when $\dot{\psi} = 0$ and hence also $W = 0$, such that

a is given by (66) and does not dependent on ψ (in fact in this case ψ does not even exist).

More general cosmologies, with ϵ in Eq. (68) being a function of z^0 are possible, provided one chooses $w(\psi)$ in Eq. (67) of a more general (non-exponential) form. In these more general cosmologies no simple relation between ψ and a exists such that Eq. (74) must be suitably generalized.

Let us have a more careful look at the power law solution (72). Recall that in the physical space an observer sees the expansion rate $\mathcal{H}$ that can be obtained from the rotated Hermitian Einstein tensor $G_{i\bar{j}}$ (53) as follows,

$$\mathcal{H}^2 = -\frac{2}{45}\Re\left(\frac{G_{i\bar{j}}}{a\bar{a}}\right) = H\bar{H}\,,$$

from which we conclude (see Eq. (72)),

$$\mathcal{H} = \frac{1}{|\epsilon|\sqrt{t^2 + (G_N E/c^4)^2}}\,. \tag{75}$$

Note that at late times $t^2 \gg G_N|E|/c^4$ the expansion rate approaches that of general relativity,

$$\mathcal{H} \to \frac{1}{|\epsilon|t}\,, \qquad (t\to\infty)\,, \tag{76}$$

with ϵ_{GR} given by $|\epsilon|$ of Hermitian gravity. In contrast to general relativity at early times $|t| \leq G_N|E|/c^4$ the expansion rate does not diverge. Instead, it reaches a maximal value at $t=0$ given by

$$\mathcal{H} \to \mathcal{H}_{\rm max} = \frac{c^4}{|\epsilon|G_N|E|} \qquad (t\to 0)\,, \tag{77}$$

which is nonsingular as long as $E \neq 0$ (below we discuss the physical relevance of the singularity at $t = 0 = E$). This behavior of $\mathcal{H}$ corresponds to a bouncing cosmology. Indeed, since the expansion rate is symmetric under time reversal, $t \to -t$, for $t < 0$ the Universe passes through a contracting phase, followed by a mirror symmetric expanding phase for $t > 0$. The time dependence of $\mathcal{H}$ on time t (on an E = const. hypersurface) is shown in figure (3). For completeness we now consider the scale factor of Hermitian cosmology. The observed scale factor $\mathcal{A}$ corresponds to the rotated metric tensor (17–18), $g_{\mu\nu} \equiv \mathcal{A}^2(x,y)\eta_{\mu\nu} = \mathrm{Re}[C_{\mu\bar{\nu}}]$. This then implies,

$$\mathcal{A} = \sqrt{a\bar{a}} = \mathcal{A}_0\left(\frac{\sqrt{t^2+(G_NE/c^4)^2}}{|\zeta_0|}\right)^{(1/\epsilon)_R} \times \tag{78}$$
$$\times \exp\left[-\left(\frac{1}{\epsilon}\right)_I\left(\mathrm{Arctan}\left(\frac{G_NE}{c^4t}\right) - \frac{\pi}{2}\mathrm{sign}(t)\right)\right],$$

where we chose $\zeta_0 = |\zeta_0|$ (the phase $\arg(\zeta_0)$ can be absorbed in $\mathcal{A}_0$), $(1/\epsilon)_R = \mathrm{Re}[\epsilon]/|\epsilon|^2$, $(1/\epsilon)_I = -\mathrm{Im}[\epsilon]/|\epsilon|^2$, and we chose the Riemann sheet of $a = a(z^0)$ such that $\mathcal{A}$ is continuous at $t=0$, as required by the equations of motion for a. The observed scale factor reaches a minimum at $t = 0$ and expands symmetrically under time reversal, for $\mathrm{Im}[\epsilon] = 0$. Nonvanishing $\mathrm{Im}[\epsilon]$, however, violates charge-parity (CP) symmetry of the ψ field, as can be seen from the potential (70). Since CPT is conserved, T must also be violated. A manifestation of this T violation can be seen in figure 4, in which we can see that the contracting phase is not a mirror image of the expanding phase. The sign of $\mathrm{Im}[\epsilon]$ determines the direction of the tilt in the scale factor function. Note that $\mathcal{H}$ *cannot* be obtained from $\mathcal{A}$ as $(\partial_t\mathcal{A})/\mathcal{A}$. [11] This should not surprise

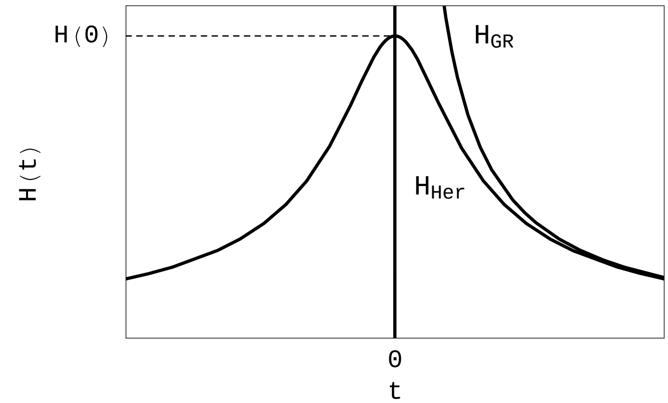


FIG. 3: The observed expansion rate as a function of time. When moving backwards in time, the expansion rate of the Hermitian Hubble parameter $\mathrm{H_{Her}}$ reaches a maximal value at $t = 0$, whereas the Hubble parameter of general relativity $\mathrm{H_{GR}}$ becomes infinite in finite time.

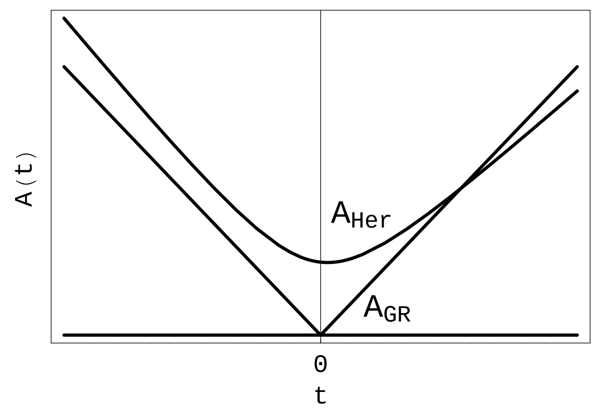


FIG. 4: The observed scale factor as a function of time. The expansion rate reaches a minimal value at $t = 0$. Nonzero $\mathrm{Im}[\epsilon]$ breaks time reversal symmetry, T.

[11] Indeed, integrating $\mathcal{H}$ would result in the scale factor proportional to

$$\left(\frac{c^4t}{G_NE} + \sqrt{\left(\frac{c^4t}{G_NE}\right)^2+1}\right)^{1/|\epsilon|},$$

which differs from Eq. (78).

us, given the fact that in Hermitian gravity a derivative of a projected quantity onto a space-time hypersurface is not in general equal to the projected derivative of the same quantity. Mathematically, the difference arises because the projection procedure must be made consistent with the Cauchy-Riemann equations.

At late times $t \gg |G_N E/c^4|$ the solution (78) approaches a power law expansion of general relativity,

$$\mathcal{A} \overset{t\to\infty}{\longrightarrow} \hat{\mathcal{A}}_0 \left(\frac{t}{|\zeta_0|}\right)^{(1/\epsilon)_R} \quad \hat{\mathcal{A}}_0 = \mathcal{A}_0 \exp\left[\frac{\pi}{2}\Big(\frac{1}{\epsilon}\Big)_I\right]. \quad (79)$$

From this we see that at late times it is difficult to distinguish the standard FLRW cosmology and Hermitian cosmology. Indeed, the only difference is in the size of the Universe: when $(1/\epsilon)_I = -\mathrm{Im}[\epsilon]/|\epsilon|^2 > 0$ ($(1/\epsilon)_I < 0$) the Universe of Hermitian cosmology appears greater (smaller) than the FLRW Universe with $\epsilon \leftrightarrow [(1/\epsilon)_R]^{-1}$. Since the absolute value of the scale factor cannot be observed (only ratios are observable), this difference cannot be used to distinguish between the standard and Hermitian cosmologies. If we had any information about the size of the early Universe, we could make the desired distinction.

One the other hand, at early times (when t and $G_N E/c^4$ are comparable), the two cosmologies differ quite dramatically. Consider first the Universe which expands such that $E = \text{const}$. In this case Eq. (78) represents a bouncing universe with a minimal size given by,

$$\mathcal{A}_{\min} = \mathcal{A}(t=0) = \mathcal{A}_0 \left(\frac{G_N|E|}{c^4|\zeta_0|}\right)^{(1/\epsilon)_R}. \quad (80)$$

The Universe behaves regularly 'everywhere' provided the cosmological singularity at $E \to 0, t \to 0$ is never reached.

A. Cosmological singularity

In order to find out how accessible the singular point of Eqs. (75) and (78) actually is, we consider a freely falling observer, which falls 'backwards in time' towards the singularity. In order to study how velocities and energy change in an expanding universe, we need to solve the corresponding geodesic equations.

Let us begin with general relativity. We are working in a spatially flat FLRW space. In conformal coordinates the Levi-Cività connection is of the form,

$$\Gamma^\mu_{\alpha\beta} = \frac{a'}{a}\Big(\delta^\mu{}_\alpha\delta_\beta{}^0 + \delta^\mu{}_\alpha\delta_\beta{}^0 + \delta^0{}_\mu\eta_{\alpha\beta}\Big).$$

This implies the following geodesic equation and line element (for a massive observer),

$$\frac{du_c^\mu}{d\tau} + \frac{a'}{a}\left(2u_c^0 u_c^\mu - \frac{\delta_0^\mu}{a^2}\right) = 0, \quad \eta_{\alpha\beta}u_c^\alpha u_c^\beta = -\frac{1}{a^2}, \quad (81)$$

where τ is the proper time observed by a freely falling observer (in the frame in which all 3-velocities vanish): $(ds)^2 = -(d\tau)^2$, and $u_c^\mu = dx_c^\mu/d\tau$ is the 4-velocity in conformal coordinates $x_c^\mu = (\eta, x_c^i)$ (here we take $c = 1$). The spatial equation (81) is easily solved,

$$\frac{d(a^2 u_c^i)}{d\eta} = 0\,, \quad (82)$$

where we made use of the definition of conformal time, $u_c^0 d\tau = d\eta$. This means that in an expanding universe, $u_c^i \propto 1/a^2$, such that the physical momentum, $p_p^i = mau_c^i$, scales as $p_p^i \propto 1/a$, where m is observer's mass. The time component of Eq. (81) implies,

$$\frac{d}{d\eta}\Big[a^2\Big(a^2(u_c^0)^2 - 1\Big)\Big] = 0\,, \quad (83)$$

which is consistent with the line element in Eq. (81) and with Eq. (82). Eqs. (81) and (82) can be also used to determine the scaling of the physical energy, $E_p \equiv p_p^0 = mau_c^0$:

$$E_p^2 - \sum_i (p_p^i)^2 = m^2\,, \quad (84)$$

from where it follows, $E_p^2 - m^2 \propto 1/a^2$ (this can be also concluded from Eq. (83)).

Let us now consider Hermitian gravity. The relevant connection coefficients are given in Eqs. (52), such that the corresponding geodesic equation (34) and the line element (12) are then,

$$\begin{aligned} \frac{du_c^\mu}{d\tau} + \frac{a'}{a}u_c^0 u_c^\mu + \frac{\bar{a}'}{\bar{a}}\Big(u_c^{\bar{0}}u_c^\mu \ + \ \eta^{\mu\bar{0}}\frac{1}{a\bar{a}}\Big) = 0 \\ \eta_{\alpha\bar{\beta}}u_c^\alpha u_c^{\bar{\beta}} \ = \ -\frac{1}{a\bar{a}} \end{aligned} \quad (85)$$

where again τ is a real affine parameter defined as the proper 'time' of a freely falling observer (in the frame in which all 3-velocities vanish and $E = 0$): $(ds)^2 = -2(d\tau)^2$, and $u_c^\mu = dz_c^\mu/d\tau$ is the complex proper 4-velocity in conformal coordinates $z_c^\mu = (z_c^0, z_c^i)$. Notice that from the definition $dz_c^0/d\tau = u_c^0$, it follows that the second term in Eq. (85) can be absorbed by a simple rescaling of u_c^μ, such that it simplifies to

$$\begin{aligned} \frac{d(au_c^\mu)}{d\tau} + \bar{H}\Big((\bar{a}u_c^{\bar{0}})(au_c^\mu) + \eta^{\mu\bar{0}}\Big) \ &= \ 0 \\ \eta_{\alpha\bar{\beta}}(au_c^\alpha)(\bar{a}u_c^{\bar{\beta}}) \ &= \ -1, \end{aligned} \quad (86)$$

where $\bar{H} = \bar{a}'/\bar{a}^2 \equiv \dot{\bar{a}}/\bar{a}$, $dz^0 = adz_c^0$ and $\dot{\bar{a}} = (1/a)da/dz_c^0$. When split into components Eq. (86) yields,

$$\frac{du^0}{d\tau} \ + \ \bar{H}\big(u^{\bar{0}}u^0 - 1\big) = 0 \quad (87a)$$

$$\frac{du^i}{d\tau} \ + \ \bar{H}u^{\bar{0}}u^i = 0\,, \qquad u^0u^{\bar{0}} - u^iu^{\bar{i}} = 1\,, \quad (87b)$$

where we defined the complex 'physical' 4-velocities $u^\mu = au_c^\mu$ and $u^{\bar\mu} = \bar a u_c^{\bar\mu}$. The corresponding complex conjugate equations must also hold. The temporal equation (87a) and its complex conjugate can be combined to give,

$$\frac{d}{d\tau}\ln\left(u^0u^{\bar 0}-1\right) = -\left(Hu^0+\bar H u^{\bar 0}\right) = -\frac{d\ln(a\bar a)}{d\tau}\,. \quad (88)$$

The last equality follows from $Hu^0 = d\ln(a)/d\tau$ and $\bar H u^{\bar 0} = d\ln(\bar a)/d\tau$. This can be straightforwardly integrated from τ_0 to τ resulting in the scaling,

$$\frac{u^0u^{\bar 0}-1}{(u^0u^{\bar 0})_0-1} = \frac{(a\bar a)_0}{a\bar a} = \frac{u^iu^{\bar i}}{(u^iu^{\bar i})_0}\,, \quad (89)$$

where the last equality follows from the constraint in Eq. (87a), or equivalently from Eq. (87b). In Eq. (89) $u^\mu = u^\mu(\tau)$, $a = a(\tau)$ and we have defined $(u^0u^{\bar 0})_0 = u^0(\tau_0)u^{\bar 0}(\tau_0)$, $(u^iu^{\bar i})_0 = u^i(\tau_0)u^{\bar i}(\tau_0)$ and $(a\bar a)_0 = a(\tau_0)\bar a(\tau_0)$. Analogously to general relativity the spatial components of particles' physical (complex) velocities scale as,

$$u^iu^{\bar i} = \left(\frac{dx^i}{d\tau}\right)^2 + G_N^2\left(\frac{dp^{\check i}}{d\tau}\right)^2 \propto \frac{1}{a\bar a} = \frac{1}{\mathcal{A}^2}\,. \quad (90)$$

Next we recall that,

$$u^0 = \frac{dz^0}{d\tau}\,, \qquad u^{\bar 0} = \frac{dz^{\bar 0}}{d\tau}$$

and we define a radial and angular (time-like) coordinates,

$$z^0 \equiv re^{i\theta} = \frac{x^0+iy^0}{\sqrt 2}\,.$$

Now, by making use of the definition $u^0 = (d/d\tau)(re^{i\theta})$ one immediately arrives at the identity,

$$\mathcal{E} \equiv \frac{1}{2}\dot r^2 + V(r,\theta) = 0\,, \qquad V = \frac{1}{2}\frac{L^2}{r^2} - \frac{1}{2}u^0u^{\bar 0}\,, \quad (91)$$

where we defined an 'angular momentum'

$$L = r^2\dot\theta\,. \quad (92)$$

This angular momentum (or more precisely the angular velocity $\omega_\theta = \dot\theta$) characterizes the rate of mixing between the time and energy coordinates in Hermitian gravity. Equation (91) represents the conserved 'energy density' of Hermitian cosmology. Indeed, since $\mathcal{E} = 0$, the energy density (91) is trivially conserved, $\dot{\mathcal{E}} = 0$. The angular momentum (92) is, however, not generally conserved, implying that the time and energy generally mix. This can be seen from the imaginary part of Eq. (87a) which – when divided by $\bar\epsilon\bar H$ and using the Hubble parameter (72) of power law expansion – yields,

$$\dot L \equiv \frac{d}{d\tau}(r^2\dot\theta) = -\frac{\epsilon_I}{|\epsilon|^2}(u^0u^{\bar 0}-1)\,. \quad (93)$$

Requiring that the derivative of energy integral (91) vanishes, one obtains the equation of motion for r, which corresponds to the real part of Eq. (87a) (divided again by $\bar\epsilon\bar H$). That means that Eq. (91) is an integral of motion and remarkably the dynamics of particles in Hermitian cosmology reduces to a study of motion in a (simple) potential given in Eq. (91).

In order to illustrate how to completely solve the geodesic equations of Hermitian cosmology, we now restrict ourselves to the simple case when $\epsilon_I = 0$ (recall that in standard FLRW cosmology ϵ is by definition a *real* parameter). In this case Eq. (93) implies that the angular momentum $L = L_0$ is conserved, $\dot L_0 = 0$ and the potential (91) acquires the simple form,

$$V = \frac{1}{2}\frac{L_0^2}{r^2} - \frac{U_0}{2r^{2/\epsilon}} - \frac{1}{2}\,, \qquad U_0 = [(u^0u^{\bar 0})_0-1](a\bar a)_0|\zeta_0|^{2/\epsilon}\,, \quad (94)$$

where $U_0 \geq 0$ parameterizes the time-like velocity at a time τ_0. Provided $\epsilon \neq 1$ this potential has an extremum V_e at the radius r_e given by,

$$\begin{aligned} V_e &= -\frac{U_0}{2}\frac{\epsilon-1}{\epsilon}\left(\frac{U_0}{\epsilon L_0^2}\right)^{\frac{1}{\epsilon-1}} - \frac{1}{2} \\ r_e &= \left(\frac{\epsilon L_0^2}{U_0}\right)^{\frac{\epsilon}{2(\epsilon-1)}} \qquad (L_0\neq 0, U_0\neq 0). \end{aligned} \quad (95)$$

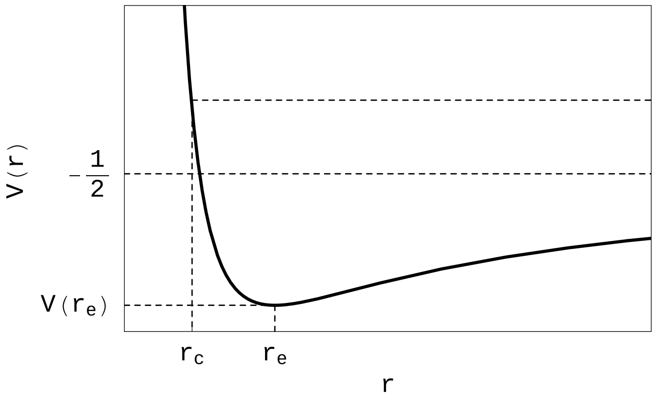
When $\epsilon > 1$ (decelerated expansion) the extremum is a minimum, as can be seen from figure 5(a). Whenever $0 < \epsilon < 1$ (accelerated expansion) the extremum is a maximum, as is depicted in figure 5(b). Choosing $\epsilon = 2$ corresponds to the radiation era and the value $\epsilon = 1/2$ is close to the ϵ parameter of today's Universe. The critical value of $\epsilon = 1$ yields a curvature dominated universe, as is shown in figure 5(c). In this case V does not have an extremum (formally, an extremum $V \to -1/2$ is reached for $r \to \infty$). As $r \to 0$ the potential approaches $+\infty$ ($-\infty$) when $L_0^2 > U_0$ ($L_0^2 < U_0$).

Since the energy integral (91) is conserved, particle dynamics in the potential (94) with $\epsilon > 1$ (or when $\epsilon = 1$ and $L_0^2 > U_0$) is such that they reach the minimal distance r_c (a turning point) given by the point where $\dot r(r_c) = 0$. In other words, the Universe of Hermitian gravity generically exhibits a bounce whenever $\epsilon > 1$ and $L_0 \neq 0$. The critical (minimal) radius is given by $V(r_c) = 0$, as can be seen from (91). In the radiation era ($\epsilon = 2$), we can solve analytically for the critical radius

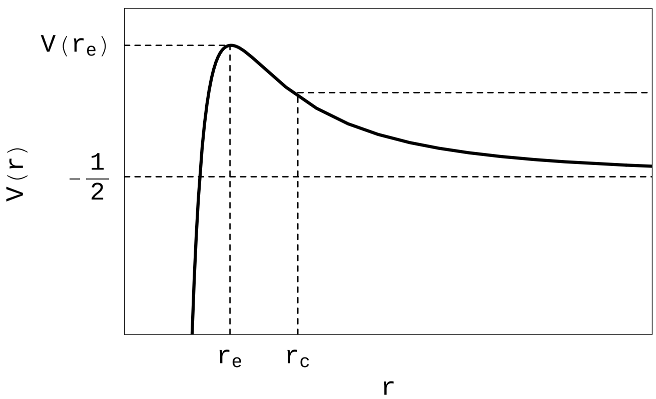
$$\begin{aligned} r_c \equiv \sqrt{t_c^2+(G_NE)^2} &= \Delta - \frac{1}{2}U_0\,, \\ \Delta = \sqrt{(U_0/2)^2+L_0^2} &\qquad (\text{radiation era})\,. \end{aligned} \quad (96)$$

The integral from of Eq. (91) is,

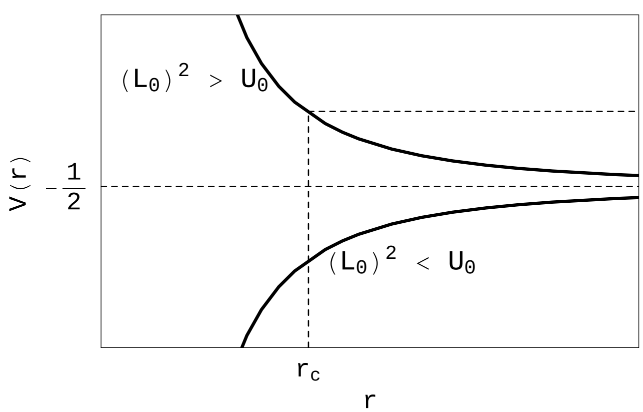
$$\int\frac{dr}{\sqrt{-2V}} = \pm\tau\,, \quad (97)$$



(a)The potential as a function of the modulus r for $\epsilon > 1$.



(b)The potential as a function of the modulus r for $0 < \epsilon < 1$. Whenever $V(r_c) > 0$ there is a bounce.



(c)The potential as a function of the modulus r for $\epsilon = 1$. For $L_0^2 > U_0$ there is a bounce.

FIG. 5: Potential (94) as a function of the radius for different values of ϵ exhibits quite generically a bounce cosmology.

which cannot be performed analytically for a general ϵ. In radiation era ($\epsilon = 2$) integrating (97) gives,

$$\begin{aligned} \pm\ \tau &= \sqrt{r^2 + U_0 r - L_0^2} \\ &- \frac{U_0}{2}\ln\left(\frac{r + (U_0/2) + \sqrt{r^2 + U_0 r - L_0^2}}{\Delta}\right), \end{aligned} \tag{98}$$

where we chose the proper time τ such that $r(\tau = 0) = r_c$. This can be inverted close to the bounce,

$$r \simeq r_c + \frac{\Delta}{2}\left(\frac{\tau}{r_c}\right)^2, \qquad (r - r_c \ll \Delta, \tau \gg r_c)$$

At late times one gets the expected linear behavior plus a logarithmic correction which characterizes Hermitian gravity,

$$r \simeq \tau + \frac{U_0}{2}\left[\ln\left(\frac{\tau}{\Delta}\right) - 1\right], \qquad (r - r_c \gg \Delta, \tau \gg r_c)\,.$$

In the limiting case when $\epsilon = 1$ (curvature domination) Eqs. (91) and (94) can be integrated to give,

$$\sqrt{r^2 + U_0 - L_0^2} = \pm\tau \tag{99}$$

such that when $U_0 < L_0^2$ there is a bounce with the minimal Hubble length given by,

$$r_c = \sqrt{L_0^2 - U_0}\,, \qquad (\epsilon = 1,\, L_0^2 > U_0)\,. \tag{100}$$

The existence of a minimal Hubble length r_c as given by Eqs. (96) and (100) means that even when time t_c is set to zero ('Big Bang'), the Universe reaches its maximal – but finite – expansion rate (75)

$$\mathcal{H}_{\max} = \frac{1}{\epsilon r_c} = \frac{1}{\epsilon\sqrt{t_c^2 + (G_N E_c/c^4)^2}}\,, \tag{101}$$

with r_c given in Eq. (96). (Even if L_0 were set to zero initially, a small nonvanishing ϵ_I would violate angular momentum conservation (93), such that we expect that $L \neq 0$ generically close to the bounce. Moreover, the choice $L_0 = 0$ represents a set of measure zero in the space of all initial conditions $\{u^\mu(\tau_0)|\eta_{\mu\bar\nu}u^\mu(\tau_0)u^{\bar\nu}(\tau_0) = -1\}$, and in this sense the condition $L_0 = 0$ is 'almost never' realized.) Equation (101) constitutes the main result of our analysis of Hermitian cosmology, according to which Hermitian cosmology is nonsingular at the classical level.

Note that Hermitian cosmology predicts r_c, but at what time, t_c, and energy, E_c, r_c is reached depends on the initial conditions embodied by U_0 and L_0. In other words: θ_c is not predicted since the corresponding angular velocity is associated with a conserved quantity L_0. To see this let us consider the evolution of the mixing angle θ,

$$\theta = \theta_0 + \int_{r_0}^{r}\frac{L dr}{r^2\sqrt{-2V}}\,. \tag{102}$$

This can be integrated for example in radiation era ($\epsilon = 2$). This is the case, since $\epsilon_I = 0$, $L = L_0$ is conserved, and the integral (102) evaluates to

$$\theta = \theta_0 - \mathrm{Arcsin}\left(\frac{\frac{L_0^2}{r} - \frac{U_0}{2}}{\Delta}\right), \qquad \Delta^2 = L_0^2 + \left(\frac{U_0}{2}\right)^2, \tag{103}$$

where we absorbed the value of the integral at r_0 into θ_0. Because of the undetermined θ_0, $\theta_c \equiv \theta(r_c)$ is indeed not predicted. Yet demanding $\theta \to 0$ when $r \to \infty$ gives $\theta_0 = -\mathrm{Arcsin}(U_0/(2\Delta))$. At the minimal radius $r = r_c$, Eq. (103) implies $\theta_c = \theta_0 - \pi/2$, such that θ_c can be anywhere between $-\pi$ and $\pi/2$, depending on L_0 and U_0. For example, in the limit when $L_0/U_0 \to 0$, $\Delta\theta \to -\pi$, while in the opposite limit when $L_0/U_0 \to \infty$, $\Delta\theta \to -\pi/2$. Note that in the latter case the Universe's expansion rate at the minimal radius r_c is completely determined by E_c.

To complete the analysis of the geodesic equation, one needs to integrate Eq. (87b). By observing that $\bar{H}u^{\bar{0}} = d\ln(\bar{a})/\tau$, one integral can be trivially performed, resulting in

$$\frac{dz^i}{d\tau} = u^i(\tau_0)\frac{\bar{a}_0}{\bar{a}(\tau)}\,.$$

This can be integrated to get $z^i = z^i(\tau)$ in special cases by making use of the dependence of the scale factor $a = a(r,\theta)$ in Eq. (78), based on which analysis of the causal structure of Hermitian cosmology can be performed. We postpone this analysis for future work.

When $\epsilon < 1$ and when $V_e < 0$ in Eq. (95) (or when $\epsilon = 1$ and $L_0^2 < U_0$) the Universe collapses towards the Big Bang singularity $r \to 0$ in a finite time. This will be the case only when the weak energy condition is violated, that is when $\rho + 3p < 0$, where ρ and p denote the energy density and pressure of the cosmological fluid, respectively. (These statements are based on the relation, $\epsilon = (3/2)(1+w)$, where $w = p/\rho$, which holds in standard FLRW cosmology.) Notice that even when $\epsilon < 1$, the Universe may exhibit a bounce, provided $V_e > 0$, or equivalently if the angular momentum is large enough, $L_0^2 > U_0^{2-\epsilon}\epsilon^{-\epsilon}(1-\epsilon)^{-(1+\epsilon)}$. In this case there is a finite barrier for a Universe to tunnel to smaller radii where $V(r) < 0$; if that happens, the Universe hits eventually the Big Bang singularity $\mathcal{H} \to \infty$. This means that inflation and bounce cosmology are not mutually incompatible.

We have thus shown that Hermitian gravity solves the problem of Big Bang singularity of Einstein's theory in a natural way.

IX. THE COSMOLOGICAL CONSTANT PROBLEM

Let us first recall Eqs. (44a–44b), which we now write as,

$$G_{\mu\nu} + C_{\mu\nu}\Lambda = 8\pi G_N T_{\mu\nu} \tag{104a}$$

$$G_{\mu\bar{\nu}} + C_{\mu\bar{\nu}}\Lambda = 8\pi G_N T_{\mu\bar{\nu}}\,. \tag{104b}$$

Now imposing the reciprocity symmetry on shell implies

$$C_{\mu\nu} = 0\,, \tag{105}$$

which means that the geometric cosmological term cannot contribute to the holomorphic equation (104a). Furthermore, as we have seen in section VIII, the reciprocity symmetry reduces the Hermitian sector (104b) to the constraints (61a–61b).

A simple proof that these constraints cannot be met unless Λ is fully compensated by a constant term in the scalar potential follows from the observation that the form of the scalar potential $V = V(\phi,\psi)$ is uniquely given by Eq. (63), with $\Omega = \pm\sqrt{4\pi G_N\alpha/3}$, $\lambda = \Lambda/(8\pi G_N)$ and $W = W(\psi,\bar{\psi})$. Note that the term $-\Lambda/(8\pi G_N)$ in the potential (63) cancels exactly the geometric cosmological constant Λ in Eq. (104b). This proof applies only to Hermitian cosmology governed by two scalar fields ϕ and ψ as described by Eqs. (37–40).

Since this is an important point, we shall now construct an alternative proof, which shows that the assumption that the late times Universe approaches a de Sitter phase with a constant expansion rate governed by some $\Lambda_{\rm eff} > 0$ leads to contradiction, resolved by requiring $\Lambda_{\rm eff} \to 0$.

Before we proceed, let us recall the standard FLRW cosmology filled with a matter with an equation of state, $w_M = p/\rho > -1$ ($\epsilon_M = (3/2)(1+w_M)$) and a cosmological term Λ. The (classical) Hubble parameter is of the form,

$$H_{GR} = \sqrt{\frac{\Lambda}{3}}\coth\left(\epsilon_M\sqrt{\frac{\Lambda}{3}}\,t\right) \tag{106a}$$

$$\dot{H}_{GR} = \frac{\Lambda}{3}\frac{\epsilon_M}{\cosh^2\left(\epsilon_M\sqrt{\frac{\Lambda}{3}}\,t\right)}\,, \tag{106b}$$

such that at late times $t \gg (1+w_M)^{-1}\sqrt{\Lambda/3}$, the expansion rate H_{GR} approaches the de Sitter attractor, $H_{GR} \to H_{dS} = \sqrt{\Lambda/3}$, and $\dot{H}_{GR} \to 0$ exponentially fast. This means that a universe filled with any matter with an equation of state with $w_M > -1$ will eventually approach the late time de Sitter attractor. This is the case, simply because the energy density in any matter fluid, with $w_M > -1$, dilutes as $\rho_M \propto 1/a^{3(1+w_M)} \propto 1/t^2$ as the Universe expands (provided w_M is constant), such that at sufficiently late times the cosmological constant necessarily dominates.

To construct an alternative proof, let us assume that at late times the Universe approaches a solution with a non-zero effective cosmological constant $\Lambda_{\rm eff}$, which yields a constant expansion rate $H \to H_{dS} = \sqrt{\Lambda_{\rm eff}/3}$ and $\dot{H} \to 0$. $\Lambda_{\rm eff}$ is not necessarily the original geometric cosmological constant, yet it must be strictly positive and $\partial_{z^0}\Lambda_{\rm eff} \to 0$ as $|z^0| \to \infty$. Firstly, from Eq. (61a) we see that as $|z^0| \to \infty$,

$$H\bar{H} = \frac{1}{9}\big(8\pi G_N V + \Lambda\big) \to \frac{\Lambda_{\rm eff}}{9} = \text{const.} \qquad (\Lambda_{\rm eff} = 8\pi G_N V_{\rm eff})\,, \tag{107}$$

or equivalently,

$$(\partial_{z^0} H)\bar{H} \to \frac{8\pi G_N}{9}\partial_{z^0} V_{\rm eff} \;\to\; 0$$
$$H\partial_{z^{\bar{0}}}\bar{H} \to \frac{8\pi G_N}{9}\partial_{z^{\bar{0}}} V_{\rm eff} \to 0 \quad (|z^0| \;\to\; \infty)\,. \tag{108}$$

Next, we multiply Eq. (61d) by $\dot{\bar{\phi}}$ and make use of Eq. (61a) to arrive at,

$$\dot{\phi}\dot{\bar{\phi}} \to -\frac{1}{3\alpha}\frac{1}{\bar{H}}\frac{\partial V_{\rm eff}}{\partial z^{\bar{0}}}\,, \tag{109}$$

where we made use of $\dot{\bar{\phi}}\partial_{\bar{\phi}} V \;\to\; \dot{\bar{\phi}}\partial_{\bar{\phi}} V_{\rm eff} \;\equiv\; \partial V_{\rm eff}/\partial z^{\bar{0}}$, with $V_{\rm eff} = \Lambda_{\rm eff}/(8\pi G_N)$. Now combining Eqs. (109) with Eq. (61b) yields,

$$\frac{\partial \ln[V_{\rm eff}]}{\partial z^{\bar{0}}} = -\frac{1}{2}\bar{H}\,. \tag{110}$$

The analogous complex conjugate equation also holds. But from Eq. (108) we know that at late times $V_{\rm eff}$ must approach a constant, and thus

$$\frac{\partial \ln[V_{\rm eff}]}{\partial z^{\bar{0}}} \to 0\,, \qquad \frac{\partial \ln[V_{\rm eff}]}{\partial z^{0}} \to 0 \qquad (|z^0| \to \infty)\,, \tag{111}$$

implying finally that at late times $\bar{H} \to 0$, which together with Eq. (107) gives,

$$H\bar{H} \to \frac{\Lambda_{\rm eff}}{9} \to 0 \qquad (|z^0| \to \infty)\,. \tag{112}$$

This completes the proof that there is no late time de Sitter attractor driven by a nonvanishing effective cosmological term $\Lambda_{\rm eff} > 0$ in Hermitian gravity (with the two scalar field action (39) used in this article).

To summarize, we have shown that the consistency of Hermitian gravity constraints requires $\Lambda_{\rm eff} = \Lambda + 8\pi G_N V_0 \;\to\; 0$, where V_0 represents the time (and energy) independent part of the scalar potential V. In other words, any cosmological term of Hermitian gravity must be fully and precisely compensated by the corresponding scalar potential.

The question is whether this holds more generally when other types of matter fields (fermions and gauge fields) are included. And moreover, what happens when quantum corrections are included. We postpone the discussion of these (important) questions for future work.

Nevertheless, note that an appropriate choice of the potential for the second scalar field ψ can lead to arbitrary (power law) expansion rate, which also includes a near exponential expansion with $\epsilon \simeq 0$. Even though this type of conformal scalar ψ matter behaves similar to a cosmological term, it is not completely identical. In fact, the choice $\epsilon = 0$ in Eq. (70) is very particular (it entails *fine tuning*), and thus does not comprise a cosmological constant problem. Let us now consider the limit $\epsilon \to 0$. The potential (69–70)

$$w(\psi,\bar{\psi}) \to w_0 \exp\left[i\omega\sqrt{(\beta/\alpha)}\,(\psi-\bar{\psi})\right] \tag{113}$$

is oscillatory (here we used $\sqrt{-1} = i$). This potential becomes exponential if $\beta/\alpha < 0$.

Note also that $\epsilon = 1/2$ ($w = -2/3$) has special relevance. This power-law accelerated expansion is realized in the absence of the second field ψ.

Let us now rewrite Eq. (72) as,

$$a = a_0\left(1 + h\epsilon z^0\right)^{1/\epsilon} \tag{114}$$

where we shifted time $z^0 \to z^0 + \zeta_0$ and we defined, $h = 1/(\epsilon\zeta_0)$. Now upon taking the limit $\epsilon \to 0$, Eq. (114) reduces to,

$$a = a_0 \exp\left(hz^0\right) \qquad (h \in \mathbb{C})\,, \tag{115}$$

representing an (exponentially expanding) complex de Sitter universe of Hermitian gravity with the complex Hubble parameter,

$$H = h\,. \tag{116}$$

This holomorphic de Sitter space must be distinguished from the de Sitter space induced by a (real) cosmological term in the Hermitian sector of the theory.

In summary, we found that, as a consequence of the reciprocity symmetry, Hermitean gravity does not admit a cosmological term at the classical level neither in the holomorphic sector nor in the Hermitean sector of the theory. Yet it does admit a holomorphic de Sitter space realized by a holomorphic scalar field (with a holomorphic kinetic term and with a suitably fine tuned exponential potential).

We have thus formulated a generalized theory of gravitation which (at the classical level) allows Minkowski space, but does not admit de Sitter space realized by a positive cosmological constant.

X. DISCUSSION

We have formulated a generalized theory of gravity on Hermitian manifolds. Given the extensive literature on complex manifolds, we summarize (and emphasize) the novel aspects of our work and compare it to existing literature:

1. Our Hermitian theory of gravity lives on a Hermitian manifold of real dimension eight. There are four space-time (x^μ) and four momentum-energy (p^μ) coordinates. The fundamental dynamical quantity of the theory is a holomorphic tetrad, which is a function of $z^\mu = x^\mu + i(G_N/c^3)p^\mu$. The tetrad transforms by means of holomorphic coordinate transformations (14), which in general mix space-time and momentum-energy. This extends and generalizes both the principle of covariance and equivalence of general relativity. We identify the reciprocity symmetry with the operation of the almost complex structure operator, which transforms $\partial/\partial x^\mu$ into $\partial/\partial y^\mu$ and $\partial/\partial y^\mu$ into $-\partial/\partial x^\mu$.

The reciprocity symmetry of a Hermitian manifold demands that the world on 'very large scales' ($\ell \gg l_{\rm Pl}$) (general relativistic limit) mirrors the world on 'very small scales' ($\ell \ll l_{\rm Pl}$) (microscopic super-Planckian world), but with the role of space-time and momentum-energy exchanged [5]. This symmetry leaves the commutation relation (2) invariant. The reciprocity symmetry implies holomorphy of the tetrad fields. Holomorphy reduces the degrees of freedom of an eight dimensional theory to the degrees of freedom of an effectively four dimensional world, as required by all known observations.

2. The eight dimensional formulation of the theory is symmetric, and yet the (four dimensional) metric contains an antisymmetric tensor, which corresponds to the imaginary part of the metric tensor $C_{\mu\bar{\nu}}$ (19), which gives rise to dynamical torsion (this still awaits a rigorous proof). Our theory differs from other dynamical theories of torsion (see for example [2]) in the leading order dynamics of torsion. It results from a theory that is projected on the space-time submanifold such that different orders in p^μ mix as a consequence of the Cauchy-Riemann equations (see remark 6 below). In this work we do not address the dynamics of the antisymmetric part of the metric, which may be of importance for example for the dark matter of the Universe [15], for spinning black holes and for the Lens-Thirring effect. Yet the fact that our Hermitian gravity theory corresponds to a 'standard' gravity theory of a symmetric metric field on an eight dimensional (Hermitian) manifold, is a strong indication that the theory of torsion within our Hermitian gravity does not suffer from the stability problems [3, 4] of – for example – the NGT of Ref. [2].

3. We define parallel transport by means of a metric compatible covariant derivative ∇_μ. Contrary to most (mathematical) literature on Hermitian manifolds [7], our covariant derivative is metric compatible, but *not* tetrad compatible (32) in the sense discussed. We consider our definition of the covariant derivative as more natural and better physically motivated, as it stems from the action principle for test particles, $S = -m\int ds$. Our covariant derivative implies 'nonstandard' Hermitian connection coefficients (36).

4. The causal structure of the theory is changed such that in the flat space limit, the space-time-momentum-energy line element is invariant under the $U(1,3)$ group (the Hermitian line element is also invariant under complex translations and hence invariant under the Hermitian generalization of the Poincaré group). The momentum-energy coordinates can be interpreted as coordinates describing non-inertial frames. The $U(1,3)$ reduces to its subgroup, the Lorentz group $SO(1,3)$, whenever observers move inertially with respect to each other; the momentum energy part of the Hermitian flat space line element vanishes. When observers move non-inertially with respect to each other, the principle of covariance of general relativity is broken, but at the same time replaced by an extended principle of covariance (namely, the flat space metric is invariant under the U(1,3) group). Yet this breaking becomes significant only in strong gravitational fields and for large momenta and energies of observers/particles, and hence does not necessarily contradict observations.

   The causal structure of the flat space limit is changed in such a way that there is a minimal time for events to be in causal contact and a maximal radius $r_{\rm max}$ for a non-local instantaneously causally related volume. The speed of light can exceed the conventional speed of light in non-inertial frames. The requirement that signals can propagate results in an upper limit on the four force squared f^2, which describes non-inertial transformations. Since there is no lower bound on f^2, there is in principle no upper limit on the group velocity, such that superluminal propagation is allowed within our theory. When the non-inertial frame of a test particle is put 'on-shell', such that the four momentum-energy squared is given by the particle's mass, $p^2 = -m^2c^2$, then $r_{\rm max} \to G_N m/c^2$ becomes one half of the Schwarzschild radius. Our analysis is based on the geodesic equation which does not take account of the self-gravity of test particles. This suggests that the above mentioned violation of causality will get hidden within the corresponding particle's black hole radius, possibly rendering any violation of causality unobservable. In conclusion, only a more proper study of this phenomenon can fully resolve the question of causality in Hermitian gravity.

5. We define an action principle for gravity and matter, where we describe the matter by two scalar fields. The pure gravity action is holomorphic in the sense that the tetrad field is a holomorphic function. The reciprocity symmetry is imposed by a constraint action, such that it is realized at the level of the equations of motion (on-shell). This assures that the Bianchi identities are satisfied. The scalar field action is covariant and built out of scalar fields that are holomorphic functions (of z^μ). One scalar field has a Hermitian kinetic term, and another a holomorphic kinetic term; the potential is the product of a holomorphic function and its anti-holomorphic counterpart. Both scalars obey the covariant stress-energy conservation law, such that the Hermitian Einstein equations with scalar matter are consistent.

6. We study the general relativistic limit of the theory,

which is realized by projecting the dynamics onto the four dimensional space-time hypersurface. An essential element in this projection are the Cauchy-Riemann equations, which are a consequence of the reciprocity (holomorphy) symmetry of the theory. The resulting projected theory is *holographic* in the sense that, having a complete knowledge of the (complex) tetrad projected onto the four dimensional space-time manifold, allows for an unambiguous reconstruction of the full eight dimensional dynamics of Hermitian gravity (the reconstruction is essentially based on the principle of analytic extension generalized to Hermitian manifolds). We find that – to leading order in momentum-energy p^μ – the geodesic equations reduce to those of general relativity. On the other hand, the (projected) dynamical (Einstein's) equations are not mutually identical even at zeroth order in p^μ; the Cauchy-Riemann equations mix different orders of p^μ. Thus in order to check the validity of our Hermitian formulation of gravity, one ought to explicitly construct and study the Hermitian analogues of *each* of the important solutions of general relativity. Only such a detailed comparison can establish the validity of Hermitian gravity, or rule it out.

7. In order to investigate whether our Hermitian gravity is a viable alternative to general relativity, we study some important aspects of Hermitian cosmology. For definiteness and simplicity, we focus on flat, homogeneous and isotropic universes which expand according the power law. This class of solutions includes most of the important cosmological solutions, including the matter era, radiation era, inflation, and – as a limit – de Sitter space. As said before, our matter is described by two scalar fields. The purpose of the scalar field with a Hermitian kinetic term is to satisfy the constraints of the Hermitian sector of the theory. This field is used to 'mark' the scale factor of the Universe. The scalar field with a holomorphic kinetic term drives the Universe's expansion. We show that at late times, when $t \gg (G_N/c^4)E$, Hermitian cosmology reduces to FLRW cosmology of general relativity, where E denotes the relevant energy scale. At early times the two theories deviate significantly. While Einstein's theory exhibits the well known Big Bang singularity, where the curvature invariants diverge, and the theory stops giving reliable predictions, our Hermitian gravity predicts a *bounce* Universe with a calculable minimal size and maximal space-time curvature. The contracting and expanding phases of an Hermitian gravity bounce can be asymmetric. This is a consequence of time reversal violation induced if the scalar field that drives the expansion violates CP (CPT is conserved). There is a caveat though: the observers which do not exhibit a mixing between the time-like and energy-like coordinates might still experience a Big Bang singularity. However, such observers are rare, and represent a negligible class of observers with very special initial conditions (mathematically speaking, the phase space corresponding to these observers is of measure zero). Moreover the time-energy rotation can be absent only in those universes where the mixing between time and energy is not dynamically generated. Yet there is no reason to presume that our Universe does not contain such a dynamical mixing.

8. Our analysis of Hermitian cosmology confirms the expectation that, even at zeroth order, Hermitian gravity differs from Einstein's gravity. The difference becomes significant, however, when space-time curvature is large, which is still in essence an untested sector of Einstein's theory. In future work we hope to investigate other aspects of the theory, whenever space-time curvature is large, such that the difference between the two theories can again become significant, e.g. various types of black hole solutions.

9. We consider the cosmological constant problem within our theory: the pure Hermitian gravity and two holomorphic scalar fields in a cosmological setting. Our analysis shows that any cosmological constant is forbidden at the classical level, thus solving the gravitational hierarchy problem within this framework. While this is a very welcome property of the theory, it is still to a large extent a mystery, and awaits a further and deeper understanding. In particular, we are interested in the question whether a link can be established between the reciprocity symmetry and the vanishing of cosmological constant. Moreover, we would like to find out whether the cosmological constant vanishes when other kinds of matter fields (in particular fermionic and gauge fields) are included. Furthermore, we would like to investigate whether our proof can be extended to include quantum effects.

10. Finally, we are of course interested in quantizing Hermitian gravity. At this stage we stress the curious fact that the commutation relations – when imposed on the space-time and momentum-energy coordinates (2) – respect the reciprocity symmetry. This is an important hint on how to quantize Hermitian gravity.

There are various other open questions which we have not addressed here. They include: (1) can violation of the principles of equivalence and covariance be observed; (2) can Hermitian gravity describe the observed inwards spiralling of the Taylor-Hulse binary pulsar; (3) does our theory meet all of the Solar system tests; (4) is the bending of light consistent with the predicted bending by general relativity; (5) can Hermitian cosmology produce cosmological perturbations consistent with observations, *etc.*

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